APPROXIMATION OF FUNCTIONS BY FOURIER-JACOBI SUMS

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Abstract

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APPROXIMATION OF FUNCTIONS BY FOURIER-JACOBI SUMS

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Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomials, orthogonal on $[-1,1]$ with weight $p(x) = (1-x)^\alpha(1+x)^\beta$, normalized by the condition $P_n^{(\alpha,\beta)}(1) = \Gamma(n+\alpha+1)/\Gamma(n+1)\Gamma(\alpha+1)$; $S_n^{(\alpha,\beta)}[f;x]$ is the n-th partial sum of the Fourier series of the function $f(x)$ in the polynomials $P_n^{(\alpha,\beta)}(x)$. As is known,

$$S_n^{(\alpha,\beta)}[f;x] = \int_{-1}^{1} f(t)K_n(t,x)p(t)\,dt,$$

where

$$K_n(t,x) = \lambda_n \frac{P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)}{t-x},$$

$$\lambda_n = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}.$$

The aim of the present note is to establish the rate of convergence of $S_n^{(\alpha,\beta)}[f;x]$ to $f(x)$ for various classes of functions.

We consider the following classes of functions, defined on the interval $[-1,1]$: W^r is the class of functions having an absolutely continuous derivative of order $r-1$ and a derivative of order $r>0$ satisfying almost everywhere the inequality $|f^{(r)}(x)| \le 1$; W_ω^{rH} is the class of functions having a continuous derivative of order $r>0$, whose modulus of continuity $\omega(f^{(r)},\delta) \le \omega(\delta)$, where $\omega(\delta)$ is a given true majorant of moduli of continuity (i.e. $\lim_{\delta\to 0}\omega(\delta) = \omega(0) = 0$ and, for $0 \le \delta_1 < \delta_2$, $0 \le \omega(\delta_2) - \omega(\delta_1) \le \omega(\delta_2 - \delta_1)$); $H^\mu = W^0 H_\mu$ is the class of functions satisfying a Lipschitz condition of order μ $(0 \le \mu \le 1)$ with constant 1; W^{rH^μ} is the class W_ω^{rH}, where $\omega(\delta) = \delta^\mu$.

Theorem 1. Let $\alpha \ge -\frac{1}{2}$, $\beta \ge -\frac{1}{2}$,

$$L_n(x) = \int_{-1}^{1} |K_n(t,x)| p(t)\, dt$$

be the Lebesgue function of the process under consideration. Then the estimate

$$L_n(x) = O(1)\left\{\ln n + \frac{n^{\alpha+1/2}}{(n\sqrt{1-x})^{\alpha+1/2}+1} + \frac{n^{\beta+1/2}}{(n\sqrt{1+x})^{\beta+1/2}+1}\right\}$$

is valid. Here $x \in [-1,1]$, and the quantity $O(1)$ depends only on α and β.

Remark 1. The order of the indicated estimate cannot be improved, since, as Rau showed (^{1}), $L_n(1) \sim n^{\alpha+1/2}$.

Remark 2. Using the well-known inequality

$$|S_n^{(\alpha,\beta)}[f;x] - f(x)| = [L_n(x)+1]E_n(f),$$

where $E_n(f)$ is the best approximation of the function $f(x)$ by algebraic polynomials of degree not exceeding n, one can find estimates for the deviation of the Fourier–Jacobi sums from the approximated functions for all possible classes of functions.

* The class H^0 consists of measurable functions f for which $|f(x)-f(y)| \leq 1$ for all x and y.

Theorem 2. The asymptotic formula holds

$$\sup_{f\in H^\mu} \left|S_n^{(\alpha,\beta)}[f;1] - f(1)\right| =$$

$$= \frac{2^{2\mu}}{\pi^{1/2}} \int_0^{\pi/2} u^\mu \sin u\, du\, \frac{\Gamma(\mu/2+\alpha/2+1/4)\Gamma(\mu/2+\beta/2+3/4)}{\Gamma(\alpha+1)\Gamma(\mu+\alpha/2+\beta/2+1)}\, n^{\alpha+1/2-\mu} + \sigma_n n^{-\mu}.$$

Here $\alpha > -1/2,\ \beta > -1/2,$

$$\sigma_n = \begin{cases} O(n^{\alpha-1/2}\ln n), & \text{if } \alpha = 1/2,\ \beta \geq -1/2 \text{ or } \alpha \geq 1/2,\ \beta = -1/2, \\ O(n^{\alpha-1/2}) + O(1) + O(n^{\alpha-\beta-1}) & \text{in the remaining cases;} \end{cases}$$

the constants entering the O-terms depend only on α and β.

Remark 1. If $-1 < \alpha \leq -1/2$, then

$$\sup_{f\in H^{\mu}}\left|S_n^{(\alpha,\beta)}[f;1]-f(1)\right|=O(n^{-\mu}).$$

Remark 2. Since

$$\sup_{f\in H^{0}}\left|S_n^{(\alpha,\beta)}[f;1]-f(1)\right|=\frac{1}{2}L_n(1),$$

from Theorem 2, for $\mu=0$, one obtains the asymptotic formula for $L_n(1)$ found by Lorch $^{(2)}$.

Theorem 3. If the function $f(x)\in W^r$, then

$$S_n^{(\alpha,\beta)}[f;1]-f(1)=A\{S_{n-r}^{(\alpha+r,\beta+r)}[f^{(r)};1]-f^{(r)}(1)\}+$$

$$+B\int_{-1}^{1}g(t)P_{n-r-1}^{(\alpha+r+2,\beta+r+1)}(t)(1-t)^{\alpha+r}(1+t)^{\beta+r+1}\,dt, \qquad (*)$$

where $\alpha>-1,\ \beta>-1$,

$$A=2^r\frac{\Gamma(\alpha+r+1)\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+r+2)}\,\frac{(n-r)!}{n!},$$

$$B=2^{-\alpha-\beta-r-2}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\,\frac{(n-r-1)!}{n!},$$

$$g(t)=f^{(r)}(1)+rf^{(r)}(t)-r(r+1)\int_0^1 f^{(r)}(1-u+tu)u^{r-1}\,du.$$

Remark 1. If $f(x)\in W_{\omega}^{rH}$, then the second term on the right-hand side of formula $(*)$ is

$$O(\omega(n^{-1})(n^{\alpha-r-1/2}+n^{-2r})),$$

when $\alpha+r\neq 1/2$, and

$$O(\omega(n^{-1})n^{2r}\ln n),$$

when $\alpha+r=1/2$. Here the constants entering the O-terms depend on α,β, and r.

Remark 2. From Theorems 2 and 3 and the preceding remark there easily follows the asymptotic formula for

$$\sup_{f\in W^rH^\mu} \left|S_n^{(\alpha,\beta)}[f;1]-f(1)\right|.$$

Theorem 4. Let $\alpha>-1,\ \beta>-1$. Then for $x\in(-1,1)$ and $r\geq 0$ one has

$$\sup_{f\in W^rH^\mu} \left|S_n^{(\alpha,\beta)}[f;x]-f(x)\right| =$$

$$=\frac{2^{\mu+1}}{\pi^2}\int_0^{\pi/2} u^\mu \sin u\,du \left(\frac{\sqrt{1-x^2}}{n}\right)^{r+\mu} \ln n + O(n^{-r-\mu}),$$

where the remainder term depends on α,β,r, and x. The estimate of the remainder term is uniform with respect to x on the interval $[-1+\varepsilon,1-\varepsilon]$.

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Note: Figure translations are in progress. See original paper for figures.