ON THE QUESTION OF THE STABILITY OF SMALL PERIODIC SOLUTIONS

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Abstract

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1. Consider the system of ordinary differential equations

$$dx/dt = f(t, x, \lambda) \tag{1}$$

with right-hand sides that are ω-periodic in t and depend on the scalar parameter λ. We shall assume that the right-hand sides possess sufficient smoothness with respect to the aggregate of variables. Throughout the paper it is assumed that system (1), for all λ, has the zero solution: $f(t, 0, \lambda) \equiv 0$. We shall be interested in the question of the existence and stability of small nonzero ω-periodic solutions of system (1) (see ([1,2])).

We shall call the number λ_0 (see ([3,4])) a **bifurcation value** of the parameter if to every $\varepsilon > 0$ there corresponds a $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for which system (1) has a nonzero ω-periodic solution whose amplitude $\max \|x(t)\|$ does not exceed ε.

Denote by $V(t, \lambda)$ the fundamental matrix (see ([5])) of the system linearized at zero,

$$dx/dt = A(t, \lambda)x. \tag{2}$$

It is easy to see (see ([3,6])) that only those values of the parameter λ for which the monodromy matrix $V(\omega, \lambda)$ has eigenvalue 1 can be bifurcation values. The converse assertion is false in the general case.

For simplicity we shall assume that 1 is an eigenvalue of the matrix $V(\omega, 0)$, and we shall be interested in the question of the existence and stability of small periodic solutions for small values of λ. In the paper it is assumed that the eigenvalue 1 corresponds to one Jordan cell of order k of the matrix $V(\omega, 0)$. In other words, it is assumed that the order of the eigenvalue 1 is equal to 1, and its multiplicity (the dimension of the root subspace) is equal to k. The case $k = 1$ has been studied in detail in ([3]); important results of a general nature on bifurcation values have been obtained in ([6]).

2. Let $e_0, e_1, \dots, e_{k-1}$ be the eigenvector and associated vectors of the matrix $V(\omega, 0)$:

$$V(\omega,0)e_0 = e_0, \quad V(\omega,0)e_1 = e_1 - e_0, \dots, \quad V(\omega,0)e_{k-1} = e_{k-1} - e_{k-2},$$

and let $g_0, g_1, \dots, g_{k-1}$ be the eigenvector and associated vectors of the adjoint matrix $V^*(\omega,0)$:

$$V^*(\omega,0)g_0 = g_0, \quad V^*(\omega,0)g_1 = g_1 - g_0, \dots, \quad V^*(\omega,0)g_{k-1} = g_{k-1} - g_{k-2}.$$

Without loss of generality (see (^{7})) one may assume the equalities

$$(e_i, g_j) = 1 \text{ for } i+j = k-1; \quad (e_i, g_j) = 0 \text{ for } i+j \neq k-1. \tag{3}$$

Denote by $x_0(t)$ the solution of the linear system (2) for $\lambda = 0$ satisfying the initial condition $x_0(0) = e_0$; by $x_1(t)$, the solution of the same system satisfying the condition $x_1(0) = e_1$.

Let us write system (1) in the form

$$dx/dt = A(t,\lambda)x + B_m(t,x,\lambda) + F(t,x,\lambda), \tag{4}$$

where $B_m(t,x,\lambda)$ contains the terms homogeneous of order m in the space variables x, and $F(t,x,\lambda)$ contains terms of higher order of smallness. Introduce the notation

$$N = \int_0^\omega (V^{-1}(\tau,0)B_m(\tau,x_0(\tau),0), g_0)\, d\tau. \tag{5}$$

If $N \neq 0$, then system (1) for $\lambda = 0$ has no small nonzero ω-periodic solutions.

The matrix $A(t,\lambda)$ may be written in the form

$$A(t,\lambda) = A(t,0) + \lambda^p A_1(t) + o(\lambda^p), \tag{6}$$

where $p > 0$ is an integer. Let

$$\alpha_0 = \int_0^\omega (V^{-1}(\tau,0)A_1(\tau)x_0(\tau), g_0)\, d\tau. \tag{7}$$

Theorem 1. Suppose that m is even and $\alpha_0 N \neq 0$. Then zero is a bifurcation value of the parameter, and for each small nonzero λ system (1) has a unique nonzero ω-periodic solution.

Theorem 2. Suppose that m is odd and p is even. Then for $\alpha_0 N > 0$ the number zero is not a bifurcation value of the parameter, while for $\alpha_0 N < 0$ it is; moreover, for small nonzero λ system (1) has exactly two small nonzero ω-periodic solutions.

Theorem 3. Suppose that m and p are odd, $\alpha_0 N \neq 0$. Then zero is a bifurcation value of the parameter; system (1) has two nonzero ω-periodic solutions for those small λ for which $\lambda\alpha_0 N < 0$, and has no nonzero small ω-periodic solutions for those λ for which $\lambda\alpha_0 N > 0$.

The proof of these theorems uses the well-known Schmidt transformation (^{8,9}) and the Newton diagram method (^{10}).

3. Theorems 1-3 can be supplemented by the following important assertion.

Theorem 4. Suppose that $\alpha_0 N \neq 0$ and $k \geq 3$. Then all nonzero small ω-periodic solutions of system (1), whose existence follows from Theorems 1-3, are Lyapunov unstable.

The question of the stability of small periodic solutions for the case $k = 1$ is considered in (^{3}).

Consider the case $k = 2$. We shall assume that all eigenvalues μ of the matrix $V(\omega, 0)$ different from 1 lie inside the circle $|\mu| < 1$. Put

$$\delta = (m-1)\alpha_0 + 2\beta_0, \tag{8}$$

where

$$\beta_0 = \frac{1}{2}\left\{ \int_0^\omega \left(V^{-1}(\tau,0)A_1(\tau)x_0(\tau) - \frac{m\alpha_0}{N}B_m(\tau, x_0(\tau); 0), g_1\right) d\tau \right.$$
$$\left. + \int_0^\omega \left(V^{-1}(\tau,0)A_1(\tau)x_1(\tau) - \frac{m\alpha_0}{N}B_m(\tau, x_0^{m-1}(\tau), x_1(\tau); 0), g_0\right) d\tau \right\}.$$

The question of the stability of the nonzero small ω-periodic solutions corresponding to small nonzero λ and existing by virtue of Theorems 1-3 is decided by the signs of the numbers $\lambda^p\alpha_0$ and $\lambda^p\delta$.

Theorem 5. The small nonzero ω-periodic solutions are stable if both numbers $\lambda^p\alpha_0$ and $\lambda^p\delta$ are negative, and unstable if at least one of these numbers is positive.

4. In this section we consider the case when the number (7) vanishes. We shall assume that

$$A(t,\lambda) = A(t,0) + \lambda^p A_1(t) + \lambda^{p_1} A_2(t) + o(\lambda^{p_1}), \tag{9}$$

where $p_1 > p$. Denote by D the matrix with elements $d_{ij} = \xi_i \eta_j$, where $e_{k-1} = (\xi_1, \dots, \xi_m)$, $g_{k-1} = (\eta_1, \dots, \eta_n)$. As is known, the matrix $I - V(\omega, 0) + D$ has the inverse Γ. Put

$$\alpha_1 = \alpha_1^{(1)} \quad \text{if } 2p_1 < p_2; \qquad \alpha_1 = \alpha_1^{(2)} \quad \text{if } 2p_1 > p_2; \qquad \alpha_1 = \alpha_1^{(1)} + \alpha_1^{(2)}$$

$$\text{if } 2p_1 = p_2,$$

where

$$\alpha_1^{(1)} = \int_0^\omega \int_0^\omega \left(V^{-1}(\tau, 0) A_1(\tau) V(\tau, 0) \Gamma V(\omega, 0) V^{-1}(s, 0) A_1(s) x_0(s), g_0\right) \, d\tau \, ds,$$

$$\alpha_1^{(2)} = \int_0^\omega \left(V^{-1}(\tau, 0) A_2(\tau) x_0(\tau), g_0\right) \, d\tau.$$

Let $q = \min(2p, p_1)$.

Theorem 6. Let $\alpha_0 = 0$. Then the assertions of Theorems 1-3 remain valid if a_0 is everywhere replaced by a_1, and p by q.

Put

$$\beta_1 = \int_0^\omega \left(V^{-1}(\tau, 0) A_1(\tau) x_1(\tau), g_0\right) \, d\tau + \int_0^\omega \left(V^{-1}(\tau, 0) A_1(\tau) x_0(\tau), g_1\right) \, d\tau.$$

Theorem 7. Let $\alpha_0 = 0$, but $a_1 \beta_1 N \neq 0$. Let $k \geq 4$. Then all nonzero small ω-periodic solutions of system (1), whose existence follows from Theorem 6, are unstable.

For $k \leq 0$ there may appear both stable and unstable small nonzero ω-periodic solutions. The corresponding criteria are cumbersome, and we do not write them out here.

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