

# SOME PROPERTIES OF GENERALIZED SOLUTIONS OF SECOND-ORDER PARABOLIC EQUATIONS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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**SOME PROPERTIES OF GENERALIZED SOLUTIONS OF SECOND-ORDER PARABOLIC EQUATIONS**

*(Presented by Academician V. I. Smirnov on 9 VII 1965)*

The paper investigates generalized solutions (g.s.) of linear and quasilinear uniformly parabolic second-order equations, to which the papers <sup>(1-6)</sup> are devoted. Here, to characterize the coefficients of the equation, a continuous scale of norms of the spaces  $L_{p,p'}(Q_T)$  is used, i.e. the norm

$$\|u\|_{p,p_1,Q_T} = \left( \int_0^T \left( \int_{\Omega} |u|^p dx \right)^{p_1/p} dt \right)^{1/p_1}, \quad (1)$$

where  $\Omega$  is some bounded domain contained in the Euclidean space  $E_n$ , and  $Q_T = \Omega \times [0, T]$ .

Let  $Q = \Omega \times [t_1, t_2]$ . By  $V_2^{1,0}(Q)$  ( $\dot{V}_2^{1,0}(Q)$ ) we denote the closure of all functions smooth in  $\bar{Q}$  (all smooth functions equal to zero near the lateral surface of  $Q$ ) in the norm

$$u\|_{V_2^{1,0}(Q)} = \left( \max_{t \in [t_1, t_2]} \int_{\Omega} u^2(x, t) dx + \iint_Q |\text{grad } u|^2 dx dt \right)^{1/2}. \quad (2)$$

Denote by  $K_{\rho} = K_{\rho}(x_0)$  the ball of radius  $\rho$  with center at the point  $x_0$  contained in  $E_n$ ; by  $Q_{\rho,\tau} = Q_{\rho,\tau}(x_0, t_0)$  the cylinder  $K_{\rho}(x_0) \times [t_0 - \tau, t_0]$ , and let  $v_+ = \max(v, 0)$ .

**Lemma 1.** *Let the numbers  $r > 0$ ,  $k' \geq 0$  be fixed, and let the function  $v(x, t)$  belong to  $V_2^{1,0}(Q_{r,r^2})$  and, for all  $k \geq k'$ ,  $\sigma \in (0, 1)$ ,  $\theta \in (0, 1)$ ,  $\rho \in [r/2, r]$ ,  $\tau \in [(r/2)^2, r^2]$ , satisfy the inequality*

$$\max_{t \in [t_0 - \tau, t_0]} r^{-n} \int_{K_{\rho - \sigma \rho}, v(t) > k} (v - k)^2 dx + r^{-n} \iint_{Q_{\rho - \sigma \rho, \tau - \theta \tau}, v > k} |\text{grad } v|^2 dx dt \leq$$

$$\begin{aligned} &\leq \gamma_0(\sigma^{-2} + \theta^{-1})r^{-(n+2)} \iint_{(Q_{\rho,\tau}, v>k)} (v-k)^2 dx dt + \\ &+ \gamma_1 k^\delta \left\{ r^{-2} \int_{t_0-\tau}^{t_0} \left[ \frac{\text{mes}(K_\rho, v(t) > k)}{\text{mes } K_r} \right]^{\alpha-2/n} dt \right\}^\beta, \end{aligned} \quad (3)$$

where  $\gamma_0 \geq 0$ ,  $\gamma_1 \geq 0$ ,  $0 \leq \delta \leq 2$ ,  $1 \leq \alpha \leq \infty$ ,  $\beta \geq 0$ ,  $\alpha\beta = 1 + \varepsilon$ ,  $\varepsilon > -1$ .\*

\* We agree that for  $\alpha = \infty$  the expression

$$\left\{ r^{-2} \int_{t_0-\tau}^{t_0} [\text{mes}(K_\rho, v(t) > k) \text{mes}^{-1} K_r]^{\alpha-2/n} dt \right\}^{(1+\varepsilon)/\alpha}$$

is understood as the limit of the written expression as  $\alpha \rightarrow \infty$ , i.e. as

$$\max_{t \in [t_0-\tau, t_0]} [\text{mes}(K_\rho, v(t) > k) \cdot \text{mes}^{-1} K_r]^{1+\varepsilon}.$$

1) Let  $-1 < \varepsilon < 0$ ,  $\delta = 0$ . Then

$$\begin{aligned} r^{-(n+2)} \text{mes}(Q_{r/2, r^2/2}, v > h) &\leq c_1 h^{-2(n+2)/n|\varepsilon|} \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right)^{2(n+2)/n|\varepsilon|}, \\ h &\geq h_0 = c_0^{(1)} \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right). \end{aligned} \quad (4)$$

2) Let  $\varepsilon = 0$ ,  $0 \leq \delta \leq 2$ , and let  $p$  be any number greater than 1. There exists a number  $d$ , depending only on  $n$ , such that for  $\gamma_1 \leq de^{-p}$  the inequality holds

$$\begin{aligned} r^{-(n+2)} \text{mes}(Q_{r/2, r^2/2}, v > h) &\leq c_2 h^{-p} \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right)^p, \\ h &\geq h_0 = c_0^{(2)} \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right). \end{aligned} \quad (5)$$

3) Let  $\varepsilon = 0$ ,  $\delta = 0$ . Then

$$\begin{aligned} r^{-(n+2)} \text{mes}(Q_{r/2, r^2/2}, v > h) &\leq c_3 \exp \left\{ -c_4 \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right)^{-1} h \right\}, \\ h &\geq h_0 = c_0^{(3)} \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right). \end{aligned} \quad (6)$$

4) Let  $\varepsilon > 0$ ,  $0 \leq \delta \leq 2$ . Then

$$\text{vrai max}_{Q_{r/2, r^2/2}} v \leq c_5 \left( r^{-(n+2)/2} \|v_+\|_{2,2, Q_{r, r^2}} + 1 \right). \quad (7)$$

In inequalities (4)–(7), the constants  $c_0^{(i)}$  ( $i \neq 2$ ) and  $c_j$  ( $j \neq 2$ ) depend only on  $n, \gamma_0, \gamma_1, k', \alpha$ , and  $\beta$ , while the constants  $c_0^{(2)}, c_2$  depend on the same quantities and on the number  $p$ .

For  $\alpha = 1 + 2/n$ , Lemma 1 was proved in (6).

Consider in  $Q_T$  the linear equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} \left( A_{ij} \frac{\partial u}{\partial x_j} + A_i u \right) + B_i \frac{\partial u}{\partial x_i} + C u + \frac{\partial}{\partial x_i} F_i + G = 0 \quad (8)$$

under the condition that

$$\nu \sum_{i=1}^n \xi_i^2 \leq A_{ij}(x, t) \xi_i \xi_j \leq \nu^{-1} \sum_{i=1}^n \xi_i^2,$$

where  $\nu > 0$ . Let

$$\vec{A} \in L_{a, a_1}(Q_T), \quad \vec{B} \in L_{b, b_1}(Q_T), \quad C \in L_{c, c_1}(Q_T), \quad \vec{F} \in L_{f, f_1}(Q_T), \\ G \in L_{g, g_1}(Q_T),$$

and moreover

$$\|\vec{A}\|_{a, a_1, Q_T} + \|\vec{B}\|_{b, b_1, Q_T} + \|C\|_{c, c_1, Q_T} \leq \mu.$$

A generalized solution of equation (8) from the class  $V_2^{1,0}(Q_T)$  ( $\dot{V}_2^{1,0}(Q_T)$ ) is a function  $u(x, t)$  belonging to  $V_2^{1,0}(Q_T)$  ( $\dot{V}_2^{1,0}(Q_T)$ ) and satisfying, for all  $t_1, t_2 \in [0, T]$  and all  $\Phi(x, t) \in \dot{W}_2^1(\Omega \times [t_1, t_2])$ , the identity

$$\int_{\Omega} u \Phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \left[ -u \frac{\partial \Phi}{\partial t} + \left( A_{ij} \frac{\partial u}{\partial x_j} + A_i u \right) \frac{\partial \Phi}{\partial x_i} + \right. \\ \left. + \left( B_i \frac{\partial u}{\partial x_i} + C u \right) \Phi + F_i \frac{\partial \Phi}{\partial x_j} + G \Phi \right] dx dt = 0.$$

**Theorem 1.** Let

$$1/a + 2/na_1 = 1/n, \quad a_1 \in [2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n, \quad c_1 \in [1, \infty), \quad n \geq 2;$$

$$f = 2, \quad f_1 = 2;$$

$$1/g + 2/ng_1 = 2/n + \frac{1}{2}, \quad g_1 \in [1, 2] \text{ for } n > 2, \quad g_1 \in [1, 2) \text{ for } n = 2.$$

Then every generalized solution of equation (8) from the class  $\dot{V}_2^{1,0}(Q_T)$  satisfies the inequality

$$\|u\|_{V_2^{1,0}(Q_T)} \leq c \left[ \|\vec{F}\|_{f, f_1, Q_T} + \|G\|_{g, g_1, Q_T} + \|u(x, 0)\|_{2, \Omega} \right],$$

where

$$c = c(n, \nu, \mu, a_1, b_1, c_1).$$

**Theorem 2.** Under the assumptions of Theorem 1 and for any  $\varphi(x) \in L_2(\Omega)$ , there exists a unique generalized solution of equation (8) from the class  $\overset{\circ}{V}_2^{1,0}(Q)$ , equal to  $\varphi(x)$  for  $t = 0$ . Every generalized solution from this class satisfies the condition

$$\|u(x, t+h) - u(x, t)\|_{2,2,Q_T} = o(h^{1/2}).$$

Denote by  $\Gamma$  the set of points lying on the lateral surface and the lower base of the cylinder  $Q_T$ . Let  $Q'$  denote an arbitrary domain contained in  $Q_T$  and at positive distance from  $\Gamma$ .

**Theorem 3.** Let

$$1/a + 2/na_1 = 1/n, \quad a_1 \in [2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n, \quad c_1 \in [1, \infty), \quad n \geq 2;$$

$$1/f + 2/nf_1 = 1/n + \sigma/2, \quad \sigma \in (0, 1), \quad f_1 \in [2, 2/\sigma] \text{ for } n > 2, \quad f_1 \in [2, 2/\sigma] \text{ for } n = 2;$$

$$1/g + 2/ng_1 = 2/n + \theta/2, \quad \theta \in (0, 1), \quad g_1 \in [1, 2/\theta] \text{ for } n > 2, \quad g_1 \in [1, 2/\theta] \text{ for } n = 2.$$

Then an arbitrary generalized solution of equation (8) from the class  $V_2^{1,0}(Q_T)$  belongs to the space  $L_{\chi, \chi_1}(Q')$ , where

$$1/\chi + 2/n\chi_1 = \lambda/2, \quad \chi_1 \in [2/\lambda, \infty] \text{ for } n > 2, \quad \chi_1 \in (2/\lambda, \infty] \text{ for } n = 2, \quad \lambda = \max(\sigma, \theta).$$

If this generalized solution is bounded on  $\Gamma$ , then it also belongs to  $L_{\chi, \chi_1}(Q_T)$  with the same exponents  $\chi$  and  $\chi_1$ .

**Theorem 4.** Suppose that, in the assumptions of Theorem 3, the exponents  $\sigma$  and  $\theta$  are equal to zero.

Then an arbitrary generalized solution of equation (8) from the class  $V_2^{1,0}(Q_T)$  belongs to the space  $L_{\chi, \chi_1}(Q')$  with arbitrary  $\chi$  and  $\chi_1$ . If this generalized solution is bounded on  $\Gamma$ , then it also belongs to  $L_{\chi, \chi_1}(Q_T)$ .

**Theorem 5.** Let

$$1/a + 2/na_1 = 1/n - \varepsilon, \quad \varepsilon > 0, \quad a_1 \in (2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n - \delta, \quad \delta > 0, \quad c_1 \in (1, \infty), \quad n \geq 2;$$

$$1/f + 2/nf_1 = 1/n, \quad f_1 \in [2, \infty] \text{ for } n > 2, \quad f_1 \in [2, \infty) \text{ for } n = 2;$$

$$1/g + 2/ng_1 = 2/n, \quad g_1 \in [1, \infty] \text{ for } n > 2, \quad g_1 \in [1, \infty) \text{ for } n = 2.$$

Then there exists a constant  $d$ , depending only on  $n, \nu, \mu$ , such that the integrals

$$E_{Q'}(u) = \iint_{Q'} \exp\{d(\|u\|_{2,2,Q_T} + 1)^{-1}|u(x,t)|\} dx dt$$

are finite for any generalized solution of equation (8) from the class  $V_2^{1,0}(Q_T)$ . If this solution is bounded on  $\Gamma$ , then the integral  $E_{Q_T}(u)$  is also finite.

**Theorem 6.** Let

$$1/a + 2/na_1 = 1/n - \varepsilon, \quad \varepsilon > 0, \quad a_1 \in (2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n + \delta, \quad \delta > 0, \quad c_1 \in (1, \infty), \quad n \geq 2;$$

$$1/f + 2/nf_1 = 1/n - \sigma, \quad \sigma > 0, \quad f_1 \in (2, \infty] \text{ for } n > 2, \quad f_1 \in (2, \infty) \text{ for } n = 2;$$

$$1/g + 2/ng_1 = 2/n - \theta, \quad \theta > 0, \quad g_1 \in (1, \infty] \text{ for } n > 2, \quad g_1 \in (1, \infty) \text{ for } n = 2.$$

Then every generalized solution of equation (8) from the class  $V_2^{1,0}(Q_T)$  is bounded in every domain  $Q'$ . If this generalized solution is bounded on  $\Gamma$ , then it will also be bounded in the whole cylinder  $Q_T$ .

**Theorem 7.** Under the assumptions of Theorem 6, an arbitrary generalized solution of equation (8) from the class  $V_2^{1,0}(Q_T)$  is Hölder continuous in any domain  $Q'$ . If, in addition, such a generalized solution is Hölder continuous on  $\Gamma$ , then it will have a bounded Hölder constant in the whole cylinder  $Q_T$ .

We now consider the quasilinear equation

$$\frac{\partial u}{\partial t} - \frac{d}{dx_i} (A_i(x, t, u, u_x)) + A(x, t, u, u_x) = 0, \quad (9)$$

in which the functions  $A_i(x, t, u, \vec{p})$  and  $A(x, t, u, \vec{p})$  are defined for  $(x, t) \in Q_T$ ,  $|u| \leq M$ , and arbitrary  $p = (p_1, \dots, p_n)$ , are continuous in  $u$  and  $p_k$ , and satisfy the conditions

$$A_i p_i \geq \nu |\vec{p}|^2 - \varphi^{(1)}, \quad |A_i| \leq \mu |\vec{p}| + \varphi^{(2)}, \quad |A| \leq \mu |\vec{p}|^2 + \varphi^{(3)}, \quad (10)$$

where  $\nu, \mu > 0$ ,  $\varphi^{(1)} \in L_{q,q_1}(Q_T)$ ,  $\varphi^{(2)} \in L_{r,r_1}(Q_T)$ ,  $\varphi^{(3)} \in L_{s,s_1}(Q_T)$ ,  $\varphi^{(i)} \geq 0$ ,  $i = 1, 2, 3$ .

**Theorem 8.** Let  $1/q + 2/nq_1 = 2/n - \varepsilon$ ,  $\varepsilon > 0$ ,  $q_1 \in (1, \infty]$  for  $n > 2$ ,  $q_1 \in (1, \infty)$  for  $n = 2$ ;  $1/r + 2/nr_1 \leq 1/n - \delta$ ,  $\delta > 0$ ,  $r_1 \in (2, \infty]$  for  $n > 2$ ,  $r_1 \in (2, \infty)$  for  $n = 2$ ;  $1/s + 2/ns_1 = 2/n - \chi$ ,  $\chi > 0$ ,  $s_1 \in (1, \infty]$ ,  $n > 2$ ,  $s_1 \in (1, \infty)$ ,  $n = 2$ .

Then any bounded generalized solution from the class  $V_2^{1,0}(Q_T)$  of equation (9)\*, satisfying conditions (10), belongs to the Hölder space  $C_{x,t}^{\alpha,\alpha/2}(Q')$  with some  $\alpha > 0$  for any domain  $Q' \subset \overline{Q'} \subset Q_T$ .

Finally, let  $u(x, t)$  be a solution of the equation

$$\frac{\partial u}{\partial t} - A_{ij}(x, t, u, u_x)u_{x_i x_j} + A(x, t, u, u_x) = 0,$$

in which the functions  $A_{ij}(x, t, u, \vec{p})$ ,  $A(x, t, u, \vec{p})$  are defined in a neighborhood of the manifold

$$\mathfrak{M} = \{(x, t) \in \overline{Q_T}, u = u(x, t), \vec{p} = u_x(x, t)\}$$

and are continuous in  $u$  and  $p$ . Suppose further that  $A_{ij}(x, t, u, p)$  are continuously differentiable with respect to  $x, u$ , and  $p_k$ , and that on the solution  $u(x, t)$  the functions  $A_{ij}$  and  $A$  satisfy the conditions

$$\nu \sum_{i=1}^n \xi_i^2 \leq A_{ij}(x, t) \xi_i \xi_j \leq \nu^{-1} \sum_{i=1}^n \xi_i^2, \quad \nu > 0,$$

$$|\partial A_{ij} / \partial p_k| \leq \mu, \quad |\partial A_{ij} / \partial u; \partial A_{ij} / \partial x_k, A| \leq \varphi(x, t), \quad \varphi(x, t) \geq 0,$$

$$\varphi \in L_{q, q_1}(Q_T), \quad 1/q + 2/nq_1 = 1/n - \varepsilon, \quad \varepsilon > 0, \quad q_1 \in (2, \infty), \quad n \geq 2.$$

**Theorem 9.** Under the assumptions made above, the Hölder norm

$$|u_x|_{C_{x,t}^{\alpha, \alpha/2}(Q')}$$

with some  $\alpha > 0$  is estimated solely in terms of  $n, \nu, \mu, q, q_1$ , the distance from  $Q'$  to  $\Gamma$ , and

$$\max_{Q_T} |u_x|.$$

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\* That is, any function  $u(x, t)$  belonging to  $V_2^{1,0}(Q_T)$ , satisfying for all  $t_1, t_2 \in [0, T]$  and bounded  $\Phi(x, t) \in W_2^1(\Omega \times [0, T])$  the integral identity

$$\int_{\Omega} u \Phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} [A_i \Phi_{x_i} + A \Phi] dx dt = 0$$

and such that  $|u(x, t)| \leq M$ .

*Note: Figure translations are in progress. See original paper for figures.*

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