MATHEMATICS

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Abstract

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TRACES OF ELLIPTIC OPERATORS ON SUBMANIFOLDS AND K-THEORY

The authors started from the following Dirichlet-Sobolev problem: let an elliptic operator $A = A_{s,t}$ be given on a closed manifold X:

$$\Gamma^s(X, E_1) \to \Gamma^t(X, E_2),$$

where E_1, E_2 are complex vector bundles; Γ^i is the Sobolev space of sections; $s > 0,\ t < 0,\ |t| \leq |s|$. For every submanifold $Y^i \subset X$ of codimension k, define the Dirichlet-Sobolev operator

$$(A, SD) : \Gamma^s(X, E_1) \to \Gamma^t(X, E_2)/R^t_Y \oplus \sum_{j \leq l^k_{s,t}} \Gamma^{s-k/2-j}(Y, i^*E_1 \otimes S^j c\bar{n}),$$

putting

$$(A, SD)[u] = [Au(\operatorname{mod} R^t_Y), i^*u).$$

Here R^{-t}_Y is the subspace in $\Gamma^t(X, E_2)$ consisting of sections concentrated on Y; S^j is the symmetric power; n is the normal bundle to Y in X; $\bar{n}$ is its conjugate; c is complexification;

$$i^* : \Gamma^s(X, E_1) \to \sum_{j \leq l^k_{s,t}} \Gamma^{s-k/2-j}(Y, i^*E_1 \otimes S^j c\bar{n})$$

is the restriction operator with normal derivatives up to order

$$l^k_{s,t} = [-t - k/2] - 1 + \operatorname{sgn}(-t - k/2 - [t - k/2])).$$

If the operator is Fredholm,* denote its (analytic) index by $I_a(A, SD)$. Put

$$I_a(A, Y) = I_a(A, SD) - I_a(A).$$

How is $I_a(A, Y)$ to be computed? (For $I_a(A)$ there is the Atiyah–Singer formula (^{1}).)

For simplicity, we shall assume that $\dim X$ is even, since the pair X, Y (with any operator) is easily reduced to the pair $(X \times S^1, Y \times S^1)$. Denote by $\operatorname{ch} A$ the Atiyah-Singer character of the operator A; T is the Todd genus; η is the tangent bundle.

Let $R_c(SO_{2q})$ be the ring of virtual representations of the group SO_{2q}, and let $\delta \in R_c(SO_{2q})$ be such an element of this ring that its restriction to $R_c(SO_{2q-1})$ is zero. Put

$$\underline{\operatorname{ch}}\delta(V) = \operatorname{ch}\delta(V)/\chi_{2q}(V),$$

where χ_{2q} is the Euler class. (This notion is correctly defined for the universal bundle and is therefore transferred to all the others.)

Theorem 1. a) If the codimension of the submanifold Y is odd, then

$$I_a(A, Y) = 0;$$

b) If the codimension of Y is even, then the formula holds

$$I_a(A, V) = -i^*(\operatorname{ch} A \cdot T(c\eta_X))\underline{\operatorname{ch}}\left\{\sum_{\substack{p \le k \\ j \le l^k_{s,t}}} (-1)^p S^j \otimes \Lambda^p \operatorname{ch}\right\}[Y],$$

where $k = \operatorname{codim} Y$;

$$l^k_{s,t} = [-t - k/2] - 1 + \operatorname{sgn}(-t - k/2 - [-t - k/2]);$$

$|s| \ge |t|$; Λ^p is the exterior-power-taking operator.

Corollary. a) If $2k > \dim X$, then $I_a(A, Y) = 0$; b) if $2k = \dim X$, then

$$I_a(A, Y) = (\operatorname{ch}^0 A)\,(\dim_c \sum S^j cn)\,(Y \circ Y).$$

Here by ch^0 we have denoted the zero-dimensional component of the Atiyah-Singer character of the operator A; $(Y \circ Y)$ is the self-intersection index of the manifold Y.

Consider in the space $\Gamma^t(X, E)$ $(t < 0)$ the subspace

$$R^t_Y \subset \Gamma^t(X, E).$$

* Necessary and sufficient conditions for the Fredholm property of such an operator were found in ([4]).

Lemma 1. There are defined operators

$$\chi_j^t : \Gamma^{t+k/2+j}(Y, i^*E \otimes S^j cn) \to R_Y^t \subset \Gamma^t(X, E'), \qquad \chi = \sum \chi_j^{(t)},$$

whose images are all homogeneous differential expressions of order j in δ-functions concentrated on Y, $j \leq l_{s,t}^k$, and t differentiations are taken in the normal directions; the image of

$$\chi = \sum \chi_j^{(t)}$$

is exactly R_Y^t.

Let

$$F = \sum_{j \leq l_{s,t}^k} F_j \subset \sum S^j cn.$$

Define the **generalized Dirichlet-Sobolev operator**

$$(A, SD_F) : \Gamma^s(X, E_1) \to \Gamma^t(X, E_2) / \operatorname{Im} \chi_F \oplus \sum_j \Gamma^{(s)}(Y, i^*E_1 \otimes \bar{F}),$$

where $\operatorname{Im} \chi_F$ is the image of the composition

$$\sum \Gamma^{t+k/2+j}(Y, i^*E_2 \otimes F_j) \to \sum \Gamma^{t+k/2+j}(Y, i^*E_2 \otimes S^j cn) \to \Gamma^t(X, E_2)$$

and $\bar{F} = \operatorname{Hom}_C(F, \mathbb{C})$.

Theorem 2. The index of the operator (A, SD_F) is computed by the formula

$$I_a(A, SD_F) = I_a(A) + I_a(A, Y; F),$$

where

$$I_a(A, Y; F) = -i^*(\operatorname{ch} AT'(cn_X)) \frac{\operatorname{ch} F}{T(\operatorname{ch})} \chi(n)[Y].$$

We indicate here one interesting case: let $F = cF_k^q(n)$, $l_{s,t}^k = \frac{k}{2}q$, with

$$F_{k_1+k_2}^q(n_1 \oplus n_2) = F_{k_1}^q(n_1) \oplus F_{k_2}^q(n_2)$$

and

$$F_1^q(\xi) = \sum_{j \leq q} \xi^j = \sum_{j \leq q} S^j \xi$$

for a one-dimensional bundle ξ.

For such an operator (A, SD_F) we have

Corollary.

$$I_a(A, Y; F_k^q) = -i^*(\mathrm{ch}\, AT(cn_X))\frac{(q+1)^{2k}}{T(\psi^{q+1}cn)}\chi(n)[Y],$$

where $\psi^p : K(X) \to K(X)$ are the Adams operations (^{2}).

Next we shall need to consider "matrix" operators (not necessarily elliptic):

$$A : \sum_j \Gamma^{s_j}(X, E_j) \to \sum_k \Gamma^{t_k}(X, E'_k),$$

where either a) all $s_j \geq 0$, all $t_k \leq 0$, or b) all $s_j \leq 0$, all $t_k \geq 0$. The operator A is a matrix with components

$$A^{jk} : \Gamma^{s_j}(X, E_j) \to \Gamma^{t_k}(X, E'_k).$$

Let $A : \Gamma^s(X, E_1) \to \Gamma^t(X, E_2)$ be an arbitrary operator of type b), and let an embedding $i : Y \subset X$, $\mathrm{codim}\, Y = 2q$, be given. We have the following operators:

$$i_n^t : \Gamma^t(X, E_2) \to \sum_{j \leq l} \Gamma^{t-q-j}(Y, i^*E_2 \otimes S^j cn) \quad \text{(restriction)},$$

$$\chi : \sum_{j \leq l} \Gamma^{s+q+j}(Y, S^j cn \otimes i^*E_1) \to \Gamma^s(X, E_1) \quad \text{(corestriction)}.$$

Put $i_{SD}(A) = i_n \circ A \circ \chi$.

Further, we shall consider only operators of type b), replacing an elliptic operator of type a) by A^{-1}(mod comp).

For a "matrix" operator

$$A : \sum_j \Gamma^{s_j}(X, E'_j) \to \sum_k \Gamma^{t_k}(X, E'_k),$$

given by the matrix $A = (A^{jk})$, $s_j \leq 0$, $t_k \geq 0$, we put

$$i_{SD}(A) = (i_{SD}A^{jk}).$$

Let now E be a real vector bundle over X of dimension $2q$. Fix the numbers s, t and denote by

$$\mathrm{Op}_s^t(X, E)$$

the set of operators of the form

$$A_{s,t}(X, E) : \sum_{j \leq l_{s,t-q}} \Gamma^{s+q+j}(X, S^j cE \otimes E_1) \to \sum_{k \leq l_{s,t-q}} \Gamma^{t-q-k}(X, S^j c\bar{E} \otimes E_2),$$

$$l_{s,t} = [\min\{|s|, |t|\}] - 1 + \mathrm{sgn}(\min(|s|, |t|) - (\min(|s|, |t|)).$$

In the case where the bundle $E = 0$, these operators are simply the $A_{s,t}$ that we considered at the beginning. We note that the Whitney sum $\oplus$ turns the set of operators $\mathrm{Op}^{s,t}(X, E)$ into a semigroup.

Lemma 2. The operator i_{SD} defines a mapping

$$i_{SD} : \mathrm{Op}^{s,t}(X, E) \longrightarrow \mathrm{Op}^{s,t}(Y, \eta \oplus i^* E).$$

Moreover, if $i_1 : Z \subset Y,\ i_2 : Y \subset X$, then

$$(i_2 \circ i_1)_{SD} = (i_1)_{SD} \circ (i^2)_{SD}.$$

Definition. We shall call an operator $A \in \mathrm{Op}^{s,t}(X, E)$ **normally elliptic** if, for all $j \leq l_{s,t} - q$, the operators

$$A_j : \sum_{k \leq j} \Gamma^{s+q+k}(X, S^k \mathbb{C}E \otimes E_1) \longrightarrow \sum_{k \leq j} \Gamma^{t-q-k}(X, S^k \mathbb{C}\overline{E} \otimes E_2),$$

defined by the principal minors of the matrix A, are elliptic.

Normally elliptic operators define a subsemigroup

$$\mathrm{Ell}^{s,t}(X, E) \subset \mathrm{Op}^{s,t}(X, E).$$

On this subsemigroup the Atiyah–Singer character is defined:

$$\mathrm{ch} : \mathrm{Ell}^{s,t}(X, E) \longrightarrow H^*(X; Q).$$

Let us define the components

$$\mathrm{ch}_k : \mathrm{Ell}^{s,t}(X, E) \longrightarrow H^*(X, Q), \quad k \leq l_{s,t} - q.$$

Namely, let $\mathrm{ch}_0 = \mathrm{ch}\, A_0$. Represent A_j in the form of a matrix

$$A_j = \begin{pmatrix} A_{j-1} & \beta_j \\ \gamma_j & \delta_j \end{pmatrix},$$

where

$$\delta_j : \Gamma^{s+q+j}(X, S^j \mathbb{C}E \otimes E_1) \longrightarrow \Gamma^{t-q-j}(X, S^j \mathbb{C}\overline{E} \otimes E_2),$$

is, in exactness, the operator A_j^j, and β_j, γ_j are rectangular matrices. From normal ellipticity it follows that the operator

$$\tilde{\delta}_j = \delta_j - \beta_j \circ \widetilde{A}_{j-1}^{-1} \circ \gamma_j$$

is elliptic. We now put

$$\mathrm{ch}_j\, A = \mathrm{ch}\, \tilde{\delta}_j, \quad j \leq l_{s,t} - q,$$

where $\widetilde{A}_{j-1}^{-1}$ is such an operator that

$$A_{j-1} \circ \widetilde{A}_{j-1}^{-1} = 1 \pmod{\mathrm{Comp}}, \quad \widetilde{A}_{j-1}^{-1} \circ A_{j-1} = 1 \pmod{\mathrm{Comp}},$$

where Comp denotes compact operators.

The following simple fact holds.

Lemma 3. The operator $A \in \mathrm{Ell}^{s,t}(X, E)$ is homotopic (by means of an absolutely universal homotopy) in the set $\mathrm{Ell}^{s,t}(X, E)$ to the diagonal operator

$$\sum_{j \le l-q} \delta_j.$$

Introduce the notation

$$\mathrm{Ch}\, A[t] = \sum_{i \ge 0} (\mathrm{ch}_i\, A) t^i \in H^*(X, Q)[t]/t^{l-q+1},$$

$$\mathrm{Ch}\, A[1] = \mathrm{ch}\, A; \qquad \mathrm{Ph}\, \xi = \sum_{i \ge 0} (\mathrm{Ch}\, S^i \xi) t^i$$

for a bundle ξ.

We note that

$$\mathrm{Ph}(\xi \oplus \eta) = \mathrm{Ph}\, \xi \cdot \mathrm{Ph}\, \eta.$$

If $Y \subset X$, $\mathrm{codim}\, Y = 2q'$, we naturally put

$$i^* : H^*(X, Q)[t]/t^{m+1} \longrightarrow H^*(Y, Q)[t]/t^{m-q'+1}, \quad m = l - q.$$

Our main Riemann–Roch theorem holds:

Main Theorem. If $A \in \mathrm{Ell}^{s,t}(X, E)$, $\dim_R E = 2q$, and an embedding $i : Y \subset X$ of codimension $2q'$ is given such that

$$i_{SD} A \in \mathrm{Ell}^{s,t}(Y, \eta \oplus i^* E),$$

then the following relations hold in $H^*(Y, Q)[t]/t^{l-q-q'+1}$:

$$1) \quad T(c\eta_Y)\, \mathrm{Ch}\, i_{SD}(A) = \left(\mathrm{ch} \sum_{p \le 2q'} (-1)^p \Lambda^p c\eta \right) \mathrm{Ph}(\eta) \circ i^*(T(c\eta_X)\, \mathrm{Ch}\, A);$$

for $E = 0$:

$$2) \quad I_a(A; Y) = I_a(i_{SD} A) = (\mathrm{Ch}\, i_{SD} A \circ T(c\eta_Y))[Y]\big|_{t=1},$$

where $\mathrm{Ph}(\xi)$, $\mathrm{Ch}\, A$, $i_{SD} A$, $\mathrm{Ell}^{s,t}(X, E)$ are defined above, and by $I_a(A, Y)$ is meant the difference of the indices of the operators

$$I_a(A^{-1}, SD) - I_a(A^{-1}),$$

where A^{-1} is an operator of type a, inverse (mod Comp) to A. The operator (A^{-1}, SD) is Fredholm under the hypotheses of the theorem.

Concluding remark. In the immediately following work by the authors (3), the most interesting types of operators are indicated, a number of examples (of group type) are examined, the geometric meaning of the Fredholm conditions established in (4) is explained, a homomorphic variant of the Riemann-Roch type theorem given here is presented, and the connection with K-theory and algebraic geometry is clarified.

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Note: Figure translations are in progress. See original paper for figures.