A CRITERION FOR THE CORRECTNESS OF A UNIFORM SPACE

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A CRITERION FOR THE CORRECTNESS OF A UNIFORM SPACE

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Here uniform spaces of the most general kind are considered. By a general uniform space we shall agree to mean a set R, in which a system $\mathfrak{U}$ of binary symmetric relations (orders of nearness) between its elements is defined.

Let $A \subset R$, $U \in \mathfrak{U}$; the U-neighborhood of the set A is the set A_U of points $x \in R$ for each of which there exists a point $y \in A$ such that xUy. In the usual way we introduce a nearness relation between nonempty subsets of R: $A\bar{\delta}B$ if and only if there exists $U \in \mathfrak{U}$ such that $A_U \cap B = \emptyset$. A general uniform space is called **correct** if it generates a nearness δ satisfying all the axioms of a nearness space (see ([1])):

0. If $A\bar{\delta}B$, then $B\bar{\delta}A$.
1. $x\bar{\delta}y$ if and only if $x \neq y$.
2. $A\bar{\delta}B \cup C$ if and only if $A\bar{\delta}B$, $A\bar{\delta}C$.
3. If $A\bar{\delta}B$, then there exist disjoint δ-neighborhoods of these sets.

Every uniform space in the sense of A. Weil is correct. On the other hand, there exist correct uniform spaces of a more general nature. Indeed, in papers ([2–4]) examples are given of nearness spaces not possessing a maximal admissible uniform structure. However, every nearness space possesses a maximal admissible pseudouniform structure ([5]). Thus pseudouniform spaces may also be correct.

V. A. Efremovich posed the problem of finding necessary and sufficient conditions for correctness of a uniform space in the most general case. The theorem proposed here gives a criterion for correctness of a general uniform space.

Theorem. *In order that a general uniform space be correct, it is necessary and sufficient that it satisfy the following three conditions:*

A. If $x, y \in R$, then $x \neq y$ if and only if there exists a relation $U \in \mathfrak{U}$ such that $x\bar{U}y$.

B. For any $M \subset R$; $U, V \in \mathfrak{U}$, there is a $W \in \mathfrak{U}$ such that

$$M_W \subset M_U \cap M_V.$$

C. For any $M \subset R$, $U \in \mathfrak{U}$, there exist $V, W \in \mathfrak{U}$ such that

$$(M_V)_W \subset M_U.$$

We note that conditions A and B were communicated to the author by V. A. Efremovich. Let us prove the necessity of the required conditions. Let (R, δ) be a nearness space and let $\mathfrak{U}$ be a general uniform structure compatible with it. From axiom 1 the necessity of condition A follows without difficulty.

Suppose that condition B is not satisfied. This means that there exist $A \subset R$ and $U, V \in \mathfrak{U}$ such that, whatever $W \in \mathfrak{U}$ is chosen, there is a point $x^W \in A_W$ such that $x^W \notin A_U \cap A_V$. The set M of all these points

can be represented in the form $M = M_1 \cup M_2$, where $A_U \cap M_1 = \emptyset$, $A_V \cap M_2 = \emptyset$. But then $A\bar{\delta}M_1$, $A\bar{\delta}M_2$ and, by axiom 2, $A\bar{\delta}M_1 \cup M_2$, i.e. $A\bar{\delta}M$. At the same time, from the construction of the set M it follows that $A\delta M$. The contradiction obtained proves the necessity of condition B.

Let $M \subset R$ and $U \in \mathfrak{u}$. Since $M_U \cap R \setminus M_U = \emptyset$, we have $\bar{M}\delta R \setminus M_U$. By axiom 3 there exist disjoint δ-neighborhoods E and F of the sets M and $R \setminus M_U$, respectively. Then $M\delta R \setminus E$, $R \setminus M_U \delta R \setminus F$; consequently, there will be $V, W \in \mathfrak{u}$ such that $M_V \cap (R \setminus E) = \emptyset$ and $(R \setminus M_U)_W \cap (R \setminus F) = \emptyset$. Hence it follows that $M_V \subset E$ and $(R \setminus M_U)_W \subset F$, but, since $E \cap F = \emptyset$, we have $M_V \cap (R \setminus M_U)_W = \emptyset$. Then $(M_V)_W \cap (R \setminus M_U) = \emptyset$, for otherwise there would be a point $x \in (M_V)_W \cap (R \setminus M_U)$. Since $x \in (M_V)_W$, there is a point $y \in M_V$ such that xWy. Hence $y \in (R \setminus M_U)_W$, i.e. $M_V \cap (R \setminus M_U)_W \neq \emptyset$, contradicting what was proved above. Thus $(M_V)_W \cap (R \setminus M_U) = \emptyset$, whence $(M_V)_W \subset M_U$, and the necessity of condition C is proved.

Let us prove sufficiency. Let the general uniform space $(R, \mathfrak{u})$ satisfy conditions A, B, C, and let the relation of proximity between its nonempty subsets be introduced in R in the usual way. The validity of axioms 0 and 1 is obvious; let us prove that axiom 2 is satisfied.

Let $A\bar{\delta}B \cup C$. This means that there exists $U \in \mathfrak{u}$ such that $A_U \cap (B \cup C) = \emptyset$. Then $A_U \cap B = \emptyset$ and $A_U \cap C = \emptyset$, i.e. $A\bar{\delta}B$ and $A\bar{\delta}C$. Conversely, if it is given that $A\bar{\delta}B$ and $A\bar{\delta}C$, then there exist $U, V \in \mathfrak{u}$ such that $A_U \cap B = \emptyset$ and $A_V \cap C = \emptyset$. By condition B there is $W \in \mathfrak{u}$ such that $A_W \subset A_U \cap A_V$; but then $A_W \cap (B \cup C) = \emptyset$, and this means precisely that $A\bar{\delta}B \cup C$. Thus it is proved that axiom 2 is satisfied.

Let $A\bar{\delta}B$. Choose $U \in \mathfrak{u}$ such that $A_U \cap B = \emptyset$. By condition C, there are $V, W \in \mathfrak{u}$ such that $(A_V)_W \subset A_U$. We assert that $A_V \cap B_W = \emptyset$. Suppose that $A_V \cap B_W \neq \emptyset$, i.e. that there is $x \in A_V \cap B_W$. Since $x \in A_V$, we have $x_W \subset (A_V)_W$, i.e. $x_W \subset A_U$. But, on the other hand, $x \in B_W$, and therefore there is a point $y \in B$ such that xWy. Then $y \in x_W$, hence $y \in A_U$, and finally $A_U \cap B \neq \emptyset$, which contradicts the supposition. Thus $A_V \cap B_W = \emptyset$, and it remains to prove that A_V and B_W are δ-neighborhoods of the sets A and B, respectively, i.e. that $A\bar{\delta}R \setminus A_V$ and $B\bar{\delta}R \setminus B_W$. But this is obvious, since $A_V \cap (R \setminus A_V) = \emptyset$ and $B_W \cap (R \setminus B_W) = \emptyset$. Thus axiom 3 is satisfied, and the proof of the theorem is thereby completely finished.

Above we have already said that there exist proximity spaces that are not uniform in A. Weil' s sense. But in the works ([2–4]), to which we referred in this connection, only the existence of such spaces is shown and no indication is given of how to construct them. It therefore seems advisable to give an example of a constructive solution of the problem posed.

Let R be the set of natural numbers, and let $\mathfrak{u}$ be the set of indices (orders of proximity) put into one-to-one correspondence with the set of those nonempty subsets of R which are either finite or are rapidly increasing sequences. Recall that a monotonically increasing sequence $\{a_n\}$ of natural numbers is called **rapidly increasing** if $\lim_{n\to\infty} a_n/n = \infty$ ([6]). For brevity, instead of "rapidly increasing sequence (finite subset $N \subset R$)," to which, by the chosen one-to-one correspondence, the index $U \in \mathfrak{u}$ corresponds, we shall say: the set of index U. Let $x, y \in R$ and $U \in \mathfrak{u}$. Put xUy if and only if x and y either both belong to, or both do not belong to, the set of index U. Let us note at once that the binary relations $U \in \mathfrak{u}$ introduced between the elements of the set R are symmetric. Let us find out what the pre-

are U-neighborhoods of nonempty subsets of the set R. Let $M \subset R$, $U \in \mathbf{u}$, and let $N \subset R$ be the index set of U. It turns out that three cases must be distinguished:

1) $M \cap N = \emptyset$. Let $x \in M_U$. Then there is a $y \in M$ such that xUy. Since $y \notin N$, by the definition of the relation U between elements of the set R, $x \notin N$. Hence $x \in R \setminus N$, and $M_U \subset R \setminus N$.

Now let $z \in R \setminus N$. Choose an arbitrary point p in M. Then z and p both do not belong to N; hence zUp, i.e. $z \in M_U$, and consequently $R \setminus N \subset M_U$.

Thus, if $M \cap N = \emptyset$, then $M_U = R \setminus N$.

2) $M \subset N$. Let $x \in M_U$. Then there is a point $y \in M$ such that xUy, but, since $M \subset N$, we have $y \in N$, and consequently $x \in N$. Hence $M_U \subset N$. Now let $z \in N$. Choose an arbitrary point p in M. Then $p \in N$ and, consequently, zUp, i.e. $z \in M_U$. Hence $N \subset M_U$ and, thus, if $M \subset N$, then $M_U = N$.

3) $M \cap N \neq \emptyset$, but $M \not\subset N$. Choose an arbitrary point $x \in R$. Then, if $x \in N$, choosing a point $y \in M \cap N$, we obtain xUy, for x and y both belong to N. Hence $x \in M_U$. If $x \notin N$, then, choosing a point $z \in M$ such that $z \notin N$, we obtain xUz, i.e. $x \in M_U$. Thus in every case $x \in M_U$; consequently, $M_U = R$.

Let us prove the correctness of the constructed general uniform space $(R, \mathbf{u})$. Let $x, y \in R$, $x \neq y$, and let $U \in \mathbf{u}$ be the relation corresponding to the singleton set $N = \{x\}$. Since $x \in N$, we have $x_U = N$, i.e. $x_U = x$. Hence $x\overline{U}y$. If, conversely, there exists $U \in \mathbf{u}$ such that $x\overline{U}y$, then $x \neq y$. This proves that $(R, \mathbf{u})$ satisfies condition A.

Let $M \subset R$; $U, V \subset \mathbf{u}$; $N_1 \subset R$ be the index set of U; $N_2 \subset R$ the index set of V. We shall prove that there exists $W \in \mathbf{u}$ such that $M_W \subset M_U \cap M_V$. As above, three cases must be considered.

1) $M \cap N_1 = \emptyset$, $M \cap N_2 = \emptyset$. Then $M_U = R \setminus N_1$, $M_V = R \setminus N_2$. Consider the set $N = N_1 \cup N_2$. It can be proved that in every case N is either finite or is a rapidly increasing sequence, i.e. it is the carrier of some index (relation) $W \in \mathbf{u}$. Since $M \cap N = \emptyset$, we have $M_W = R \setminus N$, but then $M_W \subset M_U$, $M_W \subset M_V$, i.e. $M_W \subset M_U \cap M_V$, as was required to prove.
2) Let $M \subset N_1$ (or $M \subset N_2$). Then M is the carrier of some index $W \in \mathbf{u}$. In this case $M_W = M$, hence $M_W \subset M_U \cap M_V$.
3) Let $M \cap N_1 \neq \emptyset$, but $M \not\subset N_1$. Then $M_U = R$, and one may take V as W, for $M_V \subset M_U \cap M_V$. Thus the space $(R, \mathbf{u})$ satisfies condition B.

It remains to prove that the space $(R, \mathbf{u})$ satisfies condition C. Let $M \subset R$, $U \in \mathbf{u}$, and let $N \subset R$ be the index set of U. Considering the three cases of the mutual position of M and N, it is not difficult to see that the relation $(M_U)_U = M_U$ always holds, which is stronger than condition C.

Thus, the constructed general uniform space is correct. However, it is not uniform in the sense of A. Weil. Indeed, if we assumed that $(R, \mathbf{u})$ is a uniform space, then the following condition would have to be fulfilled for it: whatever $U, V \in \mathbf{u}$ may be, there exists $W \in \mathbf{u}$ such that $x_W \subset x_U \cap x_V$ for all points $x \in R$. But this condition is not fulfilled, as may be seen in the following way. Choose arbitrary points $x, y \in R$, $x \neq y$, and suppose that $\{x\}$ is the index set of U, while $\{y\}$ is the index set of V. Then $x_U \cap x_V = \{x\}$; hence, for the chosen $U, V \in \mathbf{u}$, as $W \in \mathbf{u}$ such that the relation $x_W \subset x_U \cap x_V$ holds, one may take U and only U. On the other hand, $y_U \cap y_V = \{y\}$; hence, for the chosen U, V, as W one may take V and only V. But since $U \neq V$, one can conclude that the above condition for uniform spaces is not fulfilled in the space $(R, \mathbf{u})$.

The question of the proximity structure of the constructed correct space is of interest. If M is the index set of U, then it is far from

of any set not intersecting it. If, however, M and P are nonintersecting infinite subsets of R that cannot be arranged in rapidly increasing sequences, then $M\delta P$. For example, the set of even numbers is close to the set of odd numbers.

Of special interest are completely corrective spaces, i.e., corrective spaces satisfying the usual triangle axiom. Every completely corrective space has a unique completion; in the class of completely corrective spaces compatible with a given closeness there is a maximal element, which makes it possible to define in a natural way the completion and the product of proximity spaces.

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REFERENCES

[1] V. A. Efremovich, *Matem. sborn.*, **31** (73), 189 (1952).
[2] M. Katětov, Wiss. Zs. Humboldt Univ. Berlin, Math.-naturwiss. Kl., **9**, 685 (1959–1960).
[3] C. H. Dowker, Mappings of Proximity Structures, *General Topology and its Relations to Modern Analysis and Algebra*, Prague, 1962, pp. 139–141.
[4] E. M. Alfsen, O. Njåstad, *Fund. Mat.*, **52** (3), 235 (1963).
[5] Yu. M. Smirnov, *Tr. Mosk. matem. obshch.*, **4**, 421 (1955).
[6] P. S. Aleksandrov, P. S. Urysohn, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **31**, 3 (1950).

Note: Figure translations are in progress. See original paper for figures.