Reports of the Academy of Sciences of the USSR

Mathematical Physics

1966

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1966, Volume 166, No. 1

UDC 531.011

Mathematical Physics

F. M. IZRAILEV, B. V. CHIRIKOV

STATISTICAL PROPERTIES OF A NONLINEAR STRING

(Presented by Academician M. A. Leontovich, 3 V 1965)

The paper gives some results of an investigation of the qualitative behavior of longitudinal oscillations of a nonlinear string with fixed ends, obeying the equation:

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial z^2} \left[1 + 3\beta \left(\frac{\partial x}{\partial z} \right)^2 \right]. \quad (1)$$

This problem was studied in the work of Fermi, Pasta, and Ulam ([1]) by the method of numerical integration of the oscillations of a chain of nonlinear oscillators, approximately representing the string and obeying the system of ordinary differential equations

$$\ddot{x}_l = (x_{l+1} + x_{l-1} - 2x_l) \{1 + \beta [(x_{l+1} - x_l)^2 + (x_l - x_{l-1})^2 + $$
$$+ (x_{l+1} - x_l)(x_l - x_{l-1})]\}, \qquad l = 1, 2, \dots, N-1; \quad a = 1; \quad L = N. \quad (2)$$

The aim of work ([1]) was to trace the emergence of statistical properties in such a comparatively simple mechanical system with a large number of degrees of freedom. In the linear case ($\beta = 0$) the chain of oscillators can be represented in the form of $N - 1$ completely independent modes (normal oscillators) and, consequently, has no statistical properties (there exists a complete set of $N - 1$ single-valued integrals of motion). Until recently it was believed that any nonlinearity would lead to the appearance of statistical properties, which we shall briefly call below by the term stochasticity (ergodicity, mixing, finite entropy in the sense of A. N. Kolmogorov ([2])). Therefore the negative result of work ([1]) (a clear quasiperiodic character of the motion instead of stochasticity) seemed surprising. However, the recent works of A. N. Kolmogorov and V. I. Arnol’ d ([3],[4]) have shown that such a result is, on the contrary, natural: for a sufficiently small perturbation a nonlinear system preserves the quasiperiodic character of the motion*. From the point of view of the modern theory of dynamical systems, one should expect the existence, in the general case, of some critical perturbation at which stochasticity begins (see also ([5])). The aim of the present work is to estimate the boundary of stochasticity for the chain of oscillators ([2]).

Passing to normal coordinates (for $\beta = 0$)

$$x_l = \sqrt{\frac{2}{N-1}} \sum_{k=1}^{N-1} Q_k \sin \frac{\pi k l}{N}, \tag{3}$$

we obtain the system of equations

$$\ddot{Q}_k + \omega_k^2 Q_k = -\frac{\beta}{8N} \left\{ \sum_{i+j=2}^{k-1} A_{ij}^{+} Q_{k-i-j} \omega_{k-i-j}^2 + \right.$$

* The hypothesis of such a peculiar stability of quasiperiodic motion is contained in ([1]); in what follows we shall call it Kolmogorov stability.

$$\begin{aligned}
&+ \sum_{i+j=N-k+1}^{2N-k-1} A_{ij}^{+} Q_{2N-i-j-k} \omega_{2N-i-j-k}^2 + \sum_{i+j=2}^{N-k+1} A_{ij}^{+} Q_{i+j+k} \omega_{i+j+k}^2 \\
&- \sum_{i+j=N+k+1}^{2N-2} A_{ij}^{+} Q_{2N+k-i-j} \omega_{2N+k-i-j}^2 - \sum_{i+j=k+1}^{k+N-1} A_{ij}^{+} Q_{i+j-k} \omega_{i+j-k}^2 \\
&- \sum_{i+j=2N-k+1}^{2N-2} A_{ij}^{+} Q_{i+j+k-2N} \omega_{i+j+k-2N}^2 + 2 \sum_{i-j=k-1}^{k-N+1} A_{ij}^{-} Q_{k-i+j} \omega_{k-i+j}^2 \\
&+ 2 \sum_{i-j=N-k+1}^{N-2} A_{ij}^{-} Q_{2N-k-i+j} \omega_{2N-k-i+j}^2 - 2 \sum_{j-i=N-2}^{k+1} A_{ij}^{-} Q_{j-i-k} \omega_{j-i-k}^2
\end{aligned} \tag{4}$$

where

$$i, j, k = 1, 2, \dots, N-1; \qquad \omega_k = 2 \sin \pi k / 2N; \tag{5}$$

$$A_{ij}^{\pm} = Q_i Q_j \omega_i \omega_j \left[3\sqrt{(4-\omega_i^2)(4-\omega_j^2)} \pm \omega_i \omega_j \right]. \tag{6}$$

The expressions (4) are extremely cumbersome; therefore we shall consider two limiting cases: $k \ll N$ and $(N-k) \ll N$. Taking the perturbation to be small,

$$\beta / 8N \ll 1, \tag{7}$$

the solution of (4) can be represented in the form

$$Q_n = C_n(t)\cos\theta_n(t); \qquad \dot\theta_n = \omega'_n(t), \tag{8}$$

where $C_n,\ \omega'_n$ are slowly varying amplitudes and frequencies of the normal oscillators; the prime indicates that the frequency includes all corrections associated with the perturbation. Equations (4) can be written in the form

$$\ddot Q_k + \omega_k^2 Q_k \left\{1 - \frac{3\beta}{4N}\omega_k^2(2-\omega_k^2)Q_k^2\right\} = \frac{\beta}{8N}\sum_m F_{km}\cos\theta_{km}, \qquad \dot\theta_{km} = \omega'_{km}. \tag{9}$$

These equations describe the motion of nonlinear oscillators under the action of external forces with amplitudes $\beta F_{km}/8N$ and frequencies ω'_{km}, i.e., the case of many resonances in a nonlinear system. For a single resonance, applying the standard averaging technique ([6]), one can obtain the so-called phase equation

$$\ddot\psi_{km} = \frac{d\Omega_{km}}{dC_k}\frac{\beta F_{km}}{16\omega'_k N}\sin\psi_{km}; \qquad \Omega_{km} = \omega'_{km} - \omega'_k, \tag{10}$$

from which it is easy to find the size of the separatrix $|\dot\psi_{km}|_{\max}$, delimiting the region of stable phase oscillations near resonance (see, for example, ([7])):

$$|\dot\psi_{km}|_{\max} = \sqrt{\frac{\beta F_{km}}{4N\omega'_k}\frac{d\Omega_{km}}{dC_k}}. \tag{11}$$

In the case of many resonances, the character of the motion depends substantially on the ratio of the size of the separatrix to the mean distance between resonances $\Delta\omega$. In works ([8],[9]) it was shown that the boundary of stochasticity is determined by the condition

$$|\dot\psi_{km}|_{\max}/|\Delta\omega| \sim 1. \tag{12}$$

Let us express this condition through the dimensionless characteristic of the nonlinear perturbation (2)

$$\beta\left[(x_{l+1}-x_l)^2 + (x_l - x_{l-1})^2 + (x_{l+1}-x_l)(x_l - x_{l-1})\right] \approx 3\beta(\partial x/\partial z)^2, \tag{13}$$

since $z = la,\ a = 1$ (2). The last expression ceases to be valid—

valid for the highest modes as well; however, equality in order of magnitude is preserved.

Figure 1

Figure 1: Figure 1

Carrying out cumbersome manipulations with the right-hand side of (4), we obtain the following final estimates for the stochasticity boundary:

$$3\beta\left(\frac{\partial x}{\partial z}\right)_m^2 \sim \begin{cases} \dfrac{3}{k}, & k \ll N, \\ \dfrac{3\pi^2}{N^2}\left(\dfrac{k}{N}\right)^2, & N-k \ll N. \end{cases} \quad (14)$$

Here $(\partial x/\partial z)_m$ is the maximum value of the derivative. It is assumed that initially only one mode with number k is excited. Expression (14) is the condition for the very initial stochastic exchange between several neighboring modes. The principal difference between the two limiting cases in (14) arises from the mean distance between resonances $\Delta\omega \sim 2\pi/N$ $(k \ll N)$, $\Delta\omega \sim \pi^2/2N^2$ $(N-k \ll N)$. For a continuous string one must use the first of the estimates in (14).

Expression (14) shows that, when the lower modes are excited, stochasticity is possible only for very large nonlinear perturbations. This explains the failure of (1): a priori, excitation of the first mode seemed a quite natural initial condition. Conversely, for higher modes and large N, stochasticity begins already at very small nonlinearity.

Fig. 1. I—region of Kolmogorov stability; II—region of stochasticity; a—stochasticity boundary for $k \ll N$ (14); b—boundary for $N-k \ll N$ (14); c—qualitative interpolation; the numerical values of the straight lines a, b are given for $N = 32$; 1—result of numerical calculation for $N = 32$, $x_m = 1$, $k = 1$, $\beta = 8$ (1); 2—the same for $k = 7$, $\beta = 1/16$ (1).

In Fig. 1 the two solid straight lines show the stochasticity boundary on a double logarithmic scale, while the dashed curve represents an attempt at a rough interpolation between them; the circles are the results of numerical computation for two cases of cubic nonlinearity according to (1). It is interesting to note that the first case lies far in the region of Kolmogorov stability, despite the large value $\beta = 8$. The results of the numerical computation (1) show a clearly pronounced quasiperiodicity in this case. The second case lies near the stochasticity boundary, although the value $\beta = 1/16$ is very small, but the seventh harmonic is excited. The pattern of oscillations in this case (1) is very little like quasiperiodic motion and rather resembles undeveloped stochasticity.

We express our deep gratitude to Prof. S. M. Ulam for kindly providing report (1) and communicating additional interesting details, and to A. G. Sinai for many useful discussions.

Novosibirsk State
University

Received
28 IV 1965

REFERENCES CITED

1. E. Fermi, J. Pasta, S. Ulam, *Studies of Nonlinear Problems*, 1, Los Alamos Report LA-1940, 1955.

2. A. N. Kolmogorov, DAN, **119**, 861 (1958).

3. A. N. Kolmogorov, DAN, **98**, 527 (1954).

4. V. I. Arnold, UMN, **18**, 91 (1963).

5. N. S. Krylov, *Works on the Foundations of Statistical Physics*, USSR Academy of Sciences Press, 1950.

6. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1958.

7. B. V. Chirikov, DAN, **125**, 1015 (1959).

8. B. V. Chirikov, *Atomic Energy*, **6**, 630 (1959); B. V. Chirikov, Dissertation, Novosibirsk, 1959.

9. G. M. Zaslavsky, B. V. Chirikov, DAN, **159**, 306 (1964).

Note: Figure translations are in progress. See original paper for figures.