



Soviet-era science, translated into English

S. Ya. Gusman

1966

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Abstract

Full Text

S. Ya. Gusman

SCHOTTKY-AHLFORS FORMS AND SOME EXTREMAL PROBLEMS ON DIFFERENTIABLE MANIFOLDS

(Presented by Academician M. A. Lavrent'ev on 22 XII 1965)

Let V be an n -dimensional oriented differentiable manifold with a Riemannian metric $(^1)$, and suppose that the boundary ∂V of the manifold V consists of a finite number of differentiable $(n - 1)$ -dimensional manifolds without common points, while $\bar{V} = V \cup \partial V$ is compact. The closed manifold $W = V \cup \partial V \cup \tilde{V}$, where \tilde{V} is a second copy of V with the opposite orientation, will be called the double of the manifold V .

We denote by Γ the Hilbert space of complex forms on V for which $\|\varphi\|^2 = (\varphi, \varphi) = \int_V \varphi \wedge * \bar{\varphi} < \infty$, and by Γ^1 the linear manifold of forms of class C^∞ everywhere dense in Γ ; $\Gamma_c^1, \Gamma_e^1, \Gamma_h^1, \Gamma_0^1$ are the subsets of Γ^1 consisting respectively of closed ($d\varphi = 0$), exact ($\varphi = d\psi$), harmonic ($\Delta\varphi = \partial\delta\varphi + \delta d\varphi = 0$) forms and forms vanishing along ∂V ; $\Gamma_{h_0}^1 = \Gamma_h^1 \cap \Gamma_0^1$. By Γ_x^1 we denote the closure of Γ_x^1 in Γ , and by Γ_x^{1*} the set of forms of the form $*\varphi$, where $\varphi \in \Gamma_x^1$. The operations d, δ, Δ , and $*$ are defined in $(^1)$ (Ch. 5) and $(^2)$. It is proved there that $\Gamma_h^{1*} = \Gamma_h^1 = \Gamma_h^1$. Obviously, $\Gamma^* = \Gamma$ and $\Gamma^{1*} = \Gamma^1$. The quantity $\int_\gamma \omega$, where γ is a cycle and ω is a closed form, will be called the **period** of ω along γ . From the theorem on the weak duality of homology and cohomology $((^1), \S 22$ and $(^4), \S 4)$ it follows that a closed form is exact if and only if all its periods are equal to zero.

For forms on W we introduce one more operation, putting $\tilde{\varphi}(z) = \varphi(\tilde{z})$, where z is a point with a fixed system of local coordinates on V (or \tilde{V}), and \tilde{z} is the corresponding point with the symmetric system of coordinates on \tilde{V} (or V). Obviously, $\widetilde{\Delta\varphi} = \Delta\tilde{\varphi}$, $*\tilde{\varphi} = -\widetilde{*\varphi}$, and along ∂V $\tilde{\varphi} = \varphi$.

Let now ω be a harmonic form on W . Then $\omega \in \Gamma_c \cap \Gamma_c^*$ $(^2)$, and the form $(\omega - \tilde{\omega})/2$ belongs to Γ_{h_0} on V , while the form $(\omega + \tilde{\omega})/2$ belongs to $\Gamma_{h_0}^*$. Conversely, any form in Γ_{h_0} can be extended symmetrically to W , and for extending $\varphi \in \Gamma_{h_0}^*$ to W it suffices to extend $*\varphi$ symmetrically. If $\omega_1 \in \Gamma_{h_0}$ and $\omega_2 \in \Gamma_{h_0}^*$, then $\int_W \omega_1 \wedge *\bar{\omega}_2 = 0$. Forms on V that can be harmonically extended to W will be called **Schottky-Ahlfors forms**, and their class will be denoted by S . From the preceding arguments it follows that

Lemma 1.

$$S = \Gamma_{h0} + \Gamma_{h0}^*.$$

By S_p we denote the linear space of Schottky-Ahlfors forms of degree p . The dimension of S_p is equal to the number of generators of the p -dimensional homology group of W , by de Rham's theorem ^(1,2). If the p -dimensional homology group of W has g_p generators not homologous to cycles belonging to ∂V , and the p -dimensional homology group of ∂V has m_p generators not homologous to zero on V , then the p -dimensional homology group of W has $2g_p + m_p + m_{n-p}$ generators (g_p cycles on V , g_p cycles on \tilde{V} , m_p cycles on ∂V and m_{n-p} cycles conjugate to the cycles ∂V),

$$\dim S = 2 \sum_{p=0}^n (g_p + m_p),$$

whence

$$\dim \Gamma_{h0}^* = \dim \Gamma_{h0} = \frac{1}{2} \dim S = \sum_{p=0}^n (g_p + m_p).$$

Forms from Γ_{h0} have zero periods along cycles on ∂V , while forms from Γ_{h0}^* have zero periods along the cycles conjugate to them on W .

Lemma 2. *A form ω from Γ_{h0}^* with zero periods along all cycles on V is equal to zero.*

Proof. The periods of ω along cycles on \widehat{V} are also equal to zero; therefore $\varphi \in \Gamma_e^1$ on W , and on closed manifolds $\Gamma_h \cap \Gamma_e$ consists of the single zero ⁽²⁾.

Lemma 3. *Let $\gamma_1, \dots, \gamma_N$, where*

$$N = \sum_{p=0}^n (g_p + m_p),$$

be a basis of the homologies of V . Then in Γ_{h0}^ there exist forms $\omega_1, \dots, \omega_N$ such that*

$$\int_{\gamma_i} \omega_k = \delta_{ik}.$$

With the aid of Lemma 3 one proves the important

Lemma 4. *Let $\gamma_1, \dots, \gamma_l$, where $0 \leq l \leq N$, be homologically independent cycles on V . Then there exists one and only one form $\omega \in \Gamma_{h0}^*$ which has along γ_k*

the prescribed periods a_k ($k = 1, \dots, l$) and is orthogonal to all forms from Γ_{h0}^* having zero periods along $\gamma_1, \dots, \gamma_l$.

For the proof, complete the system $\gamma_1, \dots, \gamma_l$ to a complete system $\gamma_1, \dots, \gamma_N$ forming the homology group of V , and take a basis ω_k from Γ_{h0}^* satisfying the condition of Lemma 3. Then the desired form will be

$$\omega = \sum_{k=1}^N a_k \omega_k,$$

where a_1, \dots, a_l are prescribed, and a_{l+1}, \dots, a_N are determined from the system of equations

$$(\omega, \omega_k) = 0, \quad k = l + 1, \dots, N,$$

whose determinant is nonzero by virtue of the linear independence of the ω_k .

For the solution of extremal problems we shall also need the orthogonal decomposition $\Gamma_c^1 = \Gamma_{h0}^* + \Gamma_e^1$, which follows from Lemma 4 and the equality

$$(d\psi, \varphi) = \int_{\partial\bar{V}} d\psi \wedge *\bar{\varphi} = \int_{\partial\bar{V}} \psi \wedge *\bar{\varphi} - \int_{\bar{V}} \psi \wedge d(*\bar{\varphi}),$$

the right-hand side of which is equal to zero for $\varphi \in \Gamma_{h0}^*$. Passing to the limit, we obtain $\Gamma_c = \Gamma_{h0}^* + \Gamma_e$.

Problem 1. Determine a closed form on V with prescribed periods and minimal norm.

Problem 2. Determine a closed form on V with prescribed periods along certain cycles $\gamma_1, \dots, \gamma_l$ and minimal norm.

Problem 3. Find the best approximation of a given closed form φ in the class Γ_{h0}^* .

Theorem 1. *The solution of Problem 1 is the form from Γ_{h0}^* with the prescribed periods. The solution of Problem 3 is the form from Γ_{h0}^* having the same periods as the given closed form φ .*

Proof. Let $\varphi \in \Gamma_e$, $\omega \in \Gamma_{h0}^*$ on V , and let their periods coincide; let $\psi \in \Gamma_{h0}^*$. Then $\varphi - \omega \in \Gamma_e$:

$$(\omega, \varphi - \omega) = (\omega - \psi, \varphi - \omega) = 0;$$

$$\|\varphi\|^2 = (\varphi - \omega + \omega, \varphi - \omega + \omega) = \|\varphi - \omega\|^2 + \|\omega\|^2 \geq \|\omega\|^2,$$

$$\|\varphi - \psi\|^2 = (\varphi - \omega + \omega - \psi, \varphi - \omega + \omega - \psi) = \|\varphi - \omega\|^2 + \|\omega - \psi\|^2 \geq \|\varphi - \omega\|^2.$$

Theorem 2. *The solution of Problem 2 is the form ω from Γ_{h0}^* with the prescribed periods along the cycles $\gamma_1, \dots, \gamma_N$, orthogonal to all forms from Γ_{h0}^* with zero periods along these cycles.*

Proof. The form ω with the indicated properties exists and is unique by Lemma 4. Let $\varphi \in \Gamma_c$ and $\psi \in \Gamma_{h0}^*$ be forms with the same periods along $\gamma_1, \dots, \gamma_l$, while the periods of φ and ψ along $\gamma_{l+1}, \dots, \gamma_N$, where $\gamma_1, \dots, \gamma_N$ is a basis of homologies on V , are equal. Then

$$(\varphi - \psi, \psi) = 0 \quad \text{and} \quad (\omega, \psi - \omega) = 0,$$

whence

$$\|\varphi\|^2 = \|\varphi - \psi\|^2 + \|\psi\|^2 = \|\varphi - \psi\|^2 + \|\psi - \omega\|^2 + \|\omega\|^2 \geq \|\omega\|^2.$$

Let now the dimension n of the manifold V be odd. We shall call a form φ **pure** if $*\varphi = -\varphi$. Obviously, all nonzero pure forms are inhomogeneous. Denote the set of pure forms by Γ_r . Every form ψ in Γ whose degree is less than $n/2$ can be completed to the pure form $\varphi = \psi - *\psi$, and this correspondence is one-to-one. The sets of pure forms which thereby correspond to forms from $\Gamma_h, S, \Gamma_{h0}, \Gamma_{h0}^*, \Gamma_{he} = \Gamma_h \cap \Gamma_e$, will be denoted by $\Gamma_a, S_a, \Gamma_{a0}, \Gamma_{a0*}, \Gamma_{ae}$. In this notation

$$\Gamma_a = \Gamma_h \cap \Gamma_r.$$

Lemma 5. *A form $\omega \in \Gamma_{a0*}$ with zero periods along all cycles on V of dimension less than $n/2$ is equal to zero.*

Proof. $\omega = \psi - *\psi$, where ψ has degree* less than $n/2$, and zero periods, and is equal to zero by Lemma 2.

Lemma 6. *Let $\gamma_1, \dots, \gamma_M$,*

$$M = \sum_{p=0}^{(n-1)/2} (g_p + m_p),$$

be a basis of homologies of dimensions below $n/2$. Then in Γ_{a0} there exist forms $\omega_1, \dots, \omega_M$ such that*

$$\int_{\gamma_i} \omega_k = \delta_{ik}.$$

Lemma 7. *Let $\gamma_1, \dots, \gamma_l$, $0 \leq l \leq M$, be homologically independent cycles on V of dimension below $n/2$. Then there exists one and only one form $\omega \in \Gamma_{a0*}$ which has prescribed periods a_k along γ_k ($k = 1, \dots, l$) and is orthogonal to all forms in Γ_{a0*} having zero periods along $\gamma_1, \dots, \gamma_l$.*

Lemma 8. *There is the orthogonal decomposition*

$$\Gamma_a = \Gamma_{ae} \dot{+} \Gamma_{a0*}.$$

Proof. Let ω and φ have degree less than $n/2$ and let $\omega \in \Gamma_{h0}^*$, while $\varphi \in \Gamma_e$. Then

$$(\omega, \varphi) = 0$$

and

$$(\omega - *\omega, \varphi - *\varphi) = (\omega, \varphi) + (*\omega, *\varphi) = 0.$$

The required decomposition follows from

$$\Gamma_h = \Gamma_{he} \dot{+} \Gamma_{h0}^*,$$

and the latter from

$$\Gamma_c = \Gamma_e \dot{+} \Gamma_{h0}^*.$$

Problem 4. Among the forms in Γ_a on V , determine the form with prescribed periods along certain cycles $\gamma_1, \dots, \gamma_l$, whose dimension is less than $n/2$, and with minimal norm.

Problem 5. Find the best approximation of a given form $\varphi \in \Gamma_a$ in the class Γ_{a0*} .

Theorem 3. *The solution of Problem 4 is the form from Γ_{a0*} having the indicated periods along the cycles $\gamma_1, \dots, \gamma_l$, orthogonal to all forms from Γ_{a0*} having zero periods along $\gamma_1, \dots, \gamma_l$. The solution of Problem 5 is the form from Γ_{a0*} having along all cycles of dimension less than $n/2$ on V the same periods as the form φ .*

All the results stated remain valid if, in place of Γ , one considers the space of real forms.

The work was carried out under the supervision of Prof. L. I. Volkovskii, to whom the author expresses sincere gratitude.

Perm State University
named after A. M. Gorky

Received
4 XII 1965

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* If ψ is an inhomogeneous form, the largest of the degrees of the summands of ψ is meant.

Note: Figure translations are in progress. See original paper for figures.

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