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Abstract

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MATHEMATICS

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Let R^3 be three-dimensional Euclidean space; $S : \{x^2 + y^2 + z^2 = 1\}$ the unit sphere of the space R^3. Consider the following singular integral equation:

$$f_1(x,y,z) = [\alpha(z)(1-z^2) + z\beta(z)]u_1(x,y,z) + \\ + \int_S \frac{\alpha(z)[x(x-x_0)+y(y-y_0)] + \beta(z)(z-z_0)}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}} u_1(x_0,y_0,z_0)\,ds, \tag{1}$$

where $P = (x,y,z)$ and $Q = (x_0,y_0,z_0)$ are points of the sphere S; $\alpha(r)$, $\beta(r)$, and $f_1(x,y,z)$ are Hölder-continuous functions, $\alpha^2 + \beta^2 \neq 0$, $\beta(1) \neq 0$.

Using the equation of the sphere S and taking as independent variables on S the variables z and $\varphi = \operatorname{arctg} y/x$, we reduce equation (1) to the form

$$f(z,\varphi) = [\alpha(z)(1-z^2) + z\beta(z)]u(z,\varphi) + \\ + \frac{\alpha(z)z - \beta(z)}{2\sqrt{2}} \int_{-1}^{+1}\int_0^{2\pi} \frac{(z-z_0)u(z_0,\psi)}{(1-zz_0)^{3/2}[1-\lambda\cos(\psi-\varphi)]^{3/2}}\,d\psi\,dz_0 + \\ + \frac{\alpha(z)}{\sqrt{2}} \int_{-1}^{+1}\int_0^{2\pi} \frac{u(z_0,\psi)}{(1-zz_0)^{1/2}[1-\lambda\cos(\psi-\varphi)]^{1/2}}\,d\psi\,dz_0, \tag{2}$$

where

$$\lambda = \left[(1-z^2)(1-z_0^2)\right]^{1/2}/(1-zz_0).$$

The operator

$$T(u) = \frac{\alpha(z)}{\sqrt{2}} \int_{-1}^{+1}\int_0^{2\pi} \frac{u(z_0,\psi)}{(1-zz_0)^{1/2}[1-\lambda\cos(\psi-\varphi)]^{1/2}}\,d\psi\,dz_0$$

is completely continuous; therefore, first of all, we consider the equation

$$g(z,\varphi) = [\alpha(z)(1-z^2) + z\beta(z)]u(z,\varphi) + \\ + \frac{\alpha(z)z-\beta(z)}{2\sqrt{2}} \int_{-1}^{+1} \int_{0}^{2\pi} \frac{(z-z_0)u(z_0,\psi)}{(1-zz_0)^{3/2}[1-\lambda\cos(\psi-\varphi)]^{3/2}}\, d\psi\, dz_0. \tag{3}$$

We shall seek solutions of equations (2) and (3) in the class of Hölder-continuous functions. Since $g(z,\varphi)$ and $u(z,\varphi)$ are Hölder-continuous on the sphere S, they can be represented by uniformly convergent series of the form ([1])

$$u(z,\varphi) = \sum_{l=0}^{\infty} [u_l(z)\cos l\varphi + v_l(z)\sin l\varphi], \tag{4a}$$

$$g(z,\varphi) = \sum_{l=0}^{\infty} [f_l(z)\cos l\varphi + h_l(z)\sin l\varphi]. \tag{4b}$$

Substituting (4a) into (3), we obtain

$$[\alpha(z)(1-z^2) + z\beta(z)]u(z,\varphi) + \frac{\alpha(z)z-\beta(z)}{2\sqrt{2}} \int_{-1}^{+1} \frac{(z-z_0)\sqrt{\pi}}{(1-zz_0)^{3/2}(1-z^2)} \\ \times \sum_{l=0}^{\infty} \frac{\Gamma\left(l+\frac{3}{2}\right)}{2^{l-1}\Gamma(l+1)} \lambda^l F\left(l/2+\frac{1}{4},\ \frac{l}{2}-\frac{1}{4};\ l+1;\ \lambda^2\right) \\ \times [u_l(z_0)\cos l\varphi + v_l(z_0)\sin l\varphi]\, dz_0 = g(z,\varphi). \tag{5}$$

Equation (5) can be rewritten in the following form:

$$g(z,\varphi) = [\alpha(z)(1-z^2) + z\beta(z)]u(z,\varphi) + \\ + \frac{\alpha(z)z-\beta(z)}{2} \sum_{l=0}^{\infty} \int_{-1}^{+1} \frac{\lambda^l(1-zz_0)^{1/2}}{z-z_0} [u_l(z_0)\cos l\varphi + v_l(z_0)\sin l\varphi]\, dz_0 \\ + \int_{-1}^{+1} \int_{0}^{2\pi} H(z,z_0,\psi-\varphi)u(z_0,\psi)\, d\psi\, dz_0, \tag{6}$$

where

$$H(z,z_0,\psi-\varphi) =$$

$$= \frac{1}{z - z_0} \sum_{l=0}^{\infty} \lambda^l \left[\frac{\sqrt{\pi}\,\Gamma\left(l+\frac{3}{2}\right)}{2^{l-1}\Gamma(l+1)} F\left(l/2 + \frac{1}{4},\ l/2 - \frac{1}{4};\ l+1;\ \lambda^2\right) - 1 \right] \cos l(\psi - \varphi).$$

The operator

$$A(u) = \int_{-1}^{+1} \int_0^{2\pi} H(z, z_0, \psi - \varphi) u(z_0, \psi)\, d\psi\, dz_0$$

is completely continuous.

Consider the equation

$$g(z, \varphi) = [\alpha(z)(1 - z^2) + z\beta(z)]u(z, \varphi) +$$
$$+ \frac{\alpha(z)z - \beta(z)}{2} \sum_{l=0}^{\infty} \int_{-1}^{+1} \frac{\lambda^l (1 - zt)^{1/2}}{z - t} [u_l(t) \cos l\varphi + v_l(t) \sin l\varphi]\, dt. \tag{7}$$

Separating the variables in equation (7), we obtain

$$f_l(z) = [\alpha(z)(1 - z^2) + z\beta(z)]u_l(z) +$$

$$+ \frac{\alpha(z)z - \beta(z)}{2} \int_{-1}^{+1} \frac{\lambda^l (1 - zt)^{1/2}}{z - t} u_l(t)\, dt \tag{8}$$

(an analogous equation is also obtained for $v_l(z)$). To equation (8) the theory developed in (2) is applicable.

Introduce the following notation:

$$Q(z) = -[\alpha(z)z - \beta(z)]\pi i/2, \qquad P(z) = \alpha(z)(1 - z^2) + z\beta(z),$$

$$G(z) = \frac{P(z) - Q(z)(1 - z^2)^{1/2}}{P(z) + Q(z)(1 - z^2)^{1/2}},$$

$$P^*(z) = \frac{P(z)}{\{[P(z)]^2 - [Q(z)]^2(1 - z^2)\}},$$

$$Q^*(z) = \frac{Q(z)(1 - z^2)^{1/2}}{\{[P(z)]^2 - [Q(z)]^2(1 - z^2)\}}.$$

Choose that branch of the logarithm for which $\ln G(-1) = 0$; then we have $\ln G(+1) = 2n\pi i$, where $2n\pi$ is the increment of the argument of the function $G(z)$ on the interval $-1 \le z \le 1$, and n is an integer. Consider the integral operator

$$K^*(h) = P^*(z)h(z) - \frac{Q^*(z)Z(z)}{\pi i} \int_{-1}^{+1} \frac{h(t)}{Z(t)(t-z)}\, dt,$$

where $Z(z) = \omega(z)(z-1)^n$; $\omega(z)$ is a completely determined function depending only on P and Q; $\omega(z) \ne 0$ for $-1 \le z \le 1$.

Let $n \le 0$. From (8) we have

$$u_l(z) = K^*(f_l) + \mathcal{L}_l(u), \tag{9}$$

where

$$\mathcal{L}_l(u) = \frac{1}{\pi i} \int_{-1}^{+1} K^* \left(Q(r) \frac{\lambda^l (1-zt)^{1/2} - (1-z^2)^{1/2}}{z-t} \right) u(t)\, dt.$$

As $l \to \infty$, the operators $\mathcal{L}_l$ converge to the operator

$$A(u) = -K^* \left(\frac{Q(z)\sqrt{1-z^2}}{\pi i} \int_{-1}^{+1} \frac{u(t)}{t-z}\, dt \right) + Q^*(z)Q(z)(1-z^2)^{1/2} u(z).$$

The singular integral equation

$$f(z) = u(z) + A(u)$$

is always solvable and has a unique solution. It follows from this that equation (9), for sufficiently large l, is always solvable and has a unique solution. For the solution of equation (8) we have the representation [2]

$$u_l(z) = P(z) f_l(z) + \int_{-1}^{+1} \frac{N_l(z,t)}{t-z} f_l(t)\, dt + \sum_{i=1}^{m} C_i \omega_i(z), \tag{10}$$

where $N_l(z,t)$ is a completely determined bounded function; C_i are arbitrary constants; ω_i are solutions of the homogeneous equation (8), and, for the validity of representation (10), it is necessary and sufficient that the function $f_l(z)$ satisfy $-n+m$ orthogonality conditions

$$\int_{-1}^{+1} f_l(z)\psi_{li}(z)\,dz = 0, \qquad i = 0, 1, \dots, -n+m-1, \tag{11}$$

where $\psi_{li}(z)$ are linearly independent solutions of the adjoint homogeneous equation.

From the convergence of the operators $\mathcal{L}_l$ to the operator A, and from the fact that the equation $f(z) = u(z)+A(u)$ is always solvable and has a unique solution, it follows that the functions $N_l(z,t)$ are bounded uniformly with respect to l. Consider the series

$$N(z,t,\psi-\varphi,\tau) = \frac{1}{z-t}\sum_{l=0}^{\infty} N_l(z,t)\tau^l \cos l(\psi-\varphi), \qquad 0 \le \tau < 1.$$

The solution of equation (7) is written in the following form:

$$u(z,\varphi) = P(z)g(z,\varphi) + \lim_{\tau\to 1}\int_{-1}^{+1}\int_0^{2\pi} N(z,t,\psi-\varphi,\tau)g(t,\psi)\,d\psi\,dt, \tag{12}$$

and, in order that a solution (12) exist, it is necessary and sufficient that the function $g(z,\varphi)$ satisfy a countable set of conditions

orthogonality

$$\int_{-1}^{+1}\int_0^{2k} \mu_{lj}(z)e^{il\varphi}g(z,\varphi)\,d\varphi\,dz = 0, \tag{13}$$

$$j = 0, 1, \dots, -n+m-1; \quad l = 0, 1, 2, \dots$$

The operator

$$N(g) = \lim_{\tau\to 1}\int_{-1}^{+1}\int_0^{2\pi} N(z,t,\psi-\varphi,\tau)\,g(t,\psi)\,d\psi\,dt$$

is a bounded operator; therefore the following theorem is valid.

Theorem 1. If $n < 0$, then for the solvability of equation (3) it is necessary and sufficient that the function $g(z,\varphi)$ satisfy a countable set of orthogonality conditions. The homogeneous equation corresponding to equation (3) has no more than a finite number of linearly independent solutions. Equation (3) is normally solvable in the sense of Hausdorff.

The equation

$$h(z,\varphi) = \left[\alpha(z)(1-z^2) + z\beta(z)\right] v(z,\varphi) -$$

$$-\frac{1}{2\sqrt{2}} \int_{-1}^{+1} \int_0^{2\pi} \frac{[t\alpha(t) - \beta(t)](z-t)}{(1-tz)^{3/2}[1-\lambda\cos(\psi-\varphi)]^{3/2}}\, v(t,\psi)\, d\psi\, dt \qquad (14)$$

will be called the adjoint equation to equation (3). Equation (14) is also normally solvable in the sense of Hausdorff ([3]).

If $n > 0$ for equation (3), then for the adjoint equation (14) $n < 0$; therefore, also in the case $n > 0$, equation (3) is normally solvable in the sense of Hausdorff.

Theorem 2. Equation (3) is normally solvable in the sense of Hausdorff. For $n < 0$, for the solvability of equation (3) it is necessary and sufficient to impose on the function $g(z,\varphi)$ a countable set of orthogonality conditions; the corresponding homogeneous equation has no more than a finite number of linearly independent solutions. For $n > 0$, the homogeneous equation corresponding to equation (3) has infinitely many linearly independent solutions, and for the solvability of equation (3) it is necessary and sufficient to impose on the function $g(z,\varphi)$ no more than a finite number of orthogonality conditions. For $n = 0$, equation (3) is Fredholm.

Theorem 2 is also valid for the more general integral equation

$$h(z,\varphi) = \left[\alpha(z)(1-z^2) + z\beta(z)\right] u(z,\varphi) + T(u) +$$

$$+\int_{-1}^{+1} \int_0^{2\pi} \frac{[z\alpha(z) - \beta(z)](z-t)}{2\sqrt{2}(1-zt)^{3/2}[1-\lambda\cos(\psi-\varphi)]^{3/2}}\, u(t,\psi)\, d\psi\, dt, \qquad (15)$$

where the operator $T(u)$ is completely continuous. A special case of equation (15) is equation (2).

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