ON THE THEORY OF NETWORK SYSTEMS OF PLANNING AND CONTROL

CYBERNETICS AND CONTROL THEORY

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Abstract

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The direct use of the hypotheses of the method of estimates and plan revision for processing network models of projects having stochastic estimates for the durations of individual activities leads to a considerable error in determining the random variable ζ, expressing the completion time of all project work ([1,2]).

Replacing the hypothesis of the exclusive role of the critical path in forming ζ by the hypothesis that ζ is the maximum of random variables expressing the lengths of a certain group of the longest paths in the network, in view of the very great complexity of the analytical method ([3]), leads: a) to the necessity of using the method of statistical tests; b) to rejection of the hypothesis that project activities are small and close to infinitely subdivided; c) to the use of every possibility for aggregating activities.

All that has been said ultimately imposes more serious requirements on the distribution functions for the random variables τ, expressing the completion times of project activities. In this connection it should be noted that the system adopted as the basis in the method of estimates and plan revision, by its very nature, in principle does not permit construction of distribution functions for τ, since it is contradictory.

Indeed, this system consists of a four-parameter law of the β-distribution for each τ; three estimates characterizing this distribution: the optimistic t_1, the most probable $\hat{\tau}$, and the pessimistic t_2; and two fixed formulas for the mathematical expectation $\bar{\tau}$ and the variance $D(\tau)$:

$$\bar{\tau} = (t_1 + 4\hat{\tau} + t_2)/6, \qquad D(\tau) = (t_2 - t_1)^2/36. \tag{1}$$

Taking into account that the first and third estimates are specified quantiles ($^1/_{100}$, $^{99}/_{100}$), while the second is the mode of the distribution, we find that this system, generally speaking, is overdetermined, and hence contradictory.

In the present work a description is given of a certain model of an activity leading to a β-distribution. The estimates characterizing the distribution are analyzed, and a new definition is proposed for the principal estimates, permitting the traditional use in the system noted above and, moreover, a noncontradictory construction of the distribution function from these estimates. For the special case of research, prospecting, and design-and-development work in unexplored fields, the limiting case of the β-distribution in the form of a Γ-distribution is considered.

1. Model of an activity in network systems. Suppose that the beginning of an activity refers to the moment T_0, and that the completion of the activity is a random variable τ, varying in the interval (T_1, T_2) and possessing the following properties.

A. The entire time interval for performing the work (T_0, τ) consists of stages ρ_i $(i = 1, \dots, n)$ of duration $(T_1 - T_0)/n$, pertaining to the work proper, and of m intervals λ_k $(k = 1, \dots, m;\ 0 \le m \le n)$ of duration $(T_2 - T_1)/n$, pertaining to delays (i.e., of intervals

of the time required to identify and eliminate the disturbances, failures, difficulties that have arisen, for unplanned waiting, etc.).

B. If a difficulty has arisen on ρ_i, then after ρ_i there is necessarily one λ_k.

C. The event consisting in the fact that a difficulty has arisen at the i-th stage is determined by the i-th sample from some population.

D. Initially the unit population contains a fraction P favorable to difficulties.

E. With each stage the population increases by the amount ν, and, if difficulties arose at the preceding stage, then ν is favorable to difficulties, and otherwise is not favorable.

If by A_i^k we denote the event consisting in the fact that at the $(i+1)$-st stage a difficulty arose, provided that in the preceding i stages k difficulties arose, then

$$P(A_i^k) = (P + k\nu)/(1 + i\nu). \qquad (1 \le k \le i \le n).$$

Defining the probability density of the random variable

$$p_{m,n} = C_n^m \frac{p(p+\nu)\dots[p+(m-1)\nu](1-p)(1-p+\nu)\dots[1-p+(n-m-1)\nu]}{1(1+\nu)(1+2\nu)\dots[1+(n-1)\nu]}$$

and taking into account that $\tau = T_1 + (T_2 - T_1)x$, where $x = \lim_{n\to\infty} m/n$, with the notation $p/\nu = \alpha$, $(1-p)/\nu = \beta$, we obtain for τ the distribution function

$$F_\tau(t) = \frac{1}{K(\alpha,\beta)} \int_{T_1}^{T_2} (\tau - T_1)^{\alpha-1}(T_2-\tau)^{\beta-1}\, d\tau, \quad K(\alpha,\beta) = (T_2-T_1)^{\alpha+\beta-1} B(\alpha,\beta), \tag{2}$$

the mathematical expectation, mode, and variance:

$$\bar{\tau} = \frac{\beta T_1 + \alpha T_2}{\alpha + \beta}, \qquad \hat{\tau} = \frac{(\beta - 1)T_1 + (\alpha - 1)T_2}{\alpha + \beta - 2}, \qquad D(\tau) = \frac{\alpha\beta}{(\alpha + \beta)^2}\frac{(T_2 - T_1)^2}{\alpha + \beta - 1}. \tag{3}$$

2. Analysis of estimates characterizing the random variable τ. The mode of the distribution of τ, which has the meaning of the most probable estimate, is sufficiently clearly perceived in every task. It is very convenient to characterize τ by the quantities T_1 and T_2, which, in light of the foregoing, are purely technological parameters that in some cases can be discerned. For example, $T_1 = T_0-$ waiting, $T_2 = \infty$—research and exploratory work. Using the above-mentioned estimates t_1 and t_2 to describe τ, one must keep the following in mind:

A. The quantities t_1, t_2 must not be identified with T_1, T_2. Indeed, comparing the expressions for the variances (1) and (2), in the case of zero skewness we shall have $\alpha = \beta \geq 4$, whence, estimating the quantiles of the distribution, we obtain an error of 26%, for

$$t_1 - T_1 = T_2 - t_2 \gtrless \left(\frac{\alpha B(\alpha, \alpha)}{100}\right)^{1/\alpha} (T_2 - T_1) = 0.13(T_2 - T_1) \quad \text{for } \alpha = \beta = 4;$$

$$t_1 - T_1 = T_2 - t_2 \gtrless 0.16(T_2 - T_1) \quad \text{for } \alpha = \beta = 6. \tag{4}$$

B. The dependence of the quantiles t_1 and t_2 on the quantities $q_1 = 1/100$ and $q_2 = 99/100$ that define them is very strong; in other words, small changes in q_1 and q_2 lead to very large changes in t_1 and t_2.

C. The use of quantiles determined by very rare probabilities creates difficulties in forming estimates. Indeed, any expert more easily imagines quantiles $1/3, 2/3$ than quantiles $1/100, 99/100$, which in some cases are not even discernible.

3. Construction of a consistent system of estimates.

Consider the quantity $\tau_c = \bar{\tau} + \sigma(\bar{\tau} - \hat{\tau})$, where σ is equal to two or to a number close to it. The quantity τ_c in a certain sense characterizes the middle of the region where the values of the probability density are practically different from zero.

Examples. 1) $T_1 = 0,\ T_2 = 1,\ \alpha = 5,\ \beta = 2,\ \bar{\tau} = 0.714,\ \hat{\tau} = 0.8,\ \sigma = 1.5,\ \tau_c = 0.58$; 2) $\alpha = 9,\ \beta = 2,\ \sigma = 2,\ \tau_c = 0.687$.

Let us give a general definition: t_1 and t_2 are, respectively, the lower and upper (optimistic and pessimistic) estimates of the perceptible limits of the distribution of τ.

More concretely, we formulate this definition in the form

$$\text{I. } (t_1 + t_2)/2 = \tau_c. \qquad \text{II. } t_2 - t_1 = \mu\sqrt{D(\tau)}. \tag{5}$$

Here μ is an arbitrary constant subject to certain restrictions.

From the conditions $t_1, t_2 \in (T_1, T_2)$, $t_1 < \min\{\bar{\tau}, \hat{\tau}\}$, $t_2 > \max(\bar{\tau}, \hat{\tau}\}$, we obtain the conditions on μ:

$$\frac{|\beta-\alpha|}{\alpha+\beta-2}\sqrt{\frac{\alpha+\beta+1}{\alpha\beta}} < \mu < \frac{2}{\alpha+\beta-2}\sqrt{\frac{\alpha+\beta-1}{\alpha\beta}}\min\left\{\begin{array}{l}\alpha^2+\alpha\beta-\alpha(2+\sigma)+\beta\sigma\\ \beta^2+\alpha\beta-\beta(2+\sigma)+\alpha\sigma\end{array}\right\}.$$

In the case $\alpha = \beta$, this condition takes the form $0 < \mu < \sqrt{2\alpha+1}$, and for $\alpha \geq 2$ gives $\mu < \sqrt{20} \cong 4.47$. In the case $\beta = \infty$ we obtain $1 < \mu < 2(2+\sigma)/\sqrt{\alpha}$; if $\alpha \geq 2$, $\sigma = 2$, then $1 < \mu < 5.7$. Taking into account estimate (4), corresponding to the cases $\alpha = \beta = 4$ and $\alpha = \beta = 6$, and the condition $t_2 - t_1 = \mu\sqrt{D}$, we shall have $\mu \approx 4.44$ and $\mu \approx 4.9$.

Computing the value of μ for the normal distribution ($\alpha = \beta = \infty$), we shall have $t_2 - t_1 = 2 \cdot 2.33\sqrt{D(\tau)}$, i.e. $\mu = 4.66$.

The foregoing shows that, in the first approximation, one may put $\mu = \sqrt{20} \cong 4.47$. We note that although the formal definition of the estimates t_1 and t_2 depends on the particular values of σ and μ, it should be borne in mind that these values can always be corrected with the aid of the distribution function actually constructed from the estimates t_1 and t_2.

4. Construction of the distribution function.

The quantities t_1 and t_2, in the sense of the new definition, can retain their operational meaning as the minimum and maximum duration of the work under the most favorable and, respectively, unfavorable combination of circumstances, and make it possible to determine $\bar{\tau}$ and $D(\tau)$ by the formulas:

$$\bar{\tau} = (t_1 + 2\sigma\tau + t_2)/2(1+\sigma), \qquad D(\tau) = (t_2 - t_1)^2/\mu^2. \tag{6}$$

If, in addition to the quantities $t_1, \bar{\tau}, t_2$, one more technological parameter is given, for example T_1, then, analyzing expression (3), one can obtain the values for α, β, T_2 in the form

$$\alpha = (M - 1 - 2M\Delta)/[2 - (3+M)\Delta], \qquad \beta = \alpha(1+\alpha)/(M-\alpha),$$

$$T_2 = [(\alpha+\beta)\bar{\tau} - \beta T_1]/\alpha,$$

where

$$M = (\bar{\tau} - T_1)^2 / D(\tau), \qquad \Delta = (\bar{\tau} - \hat{\tau})/(\bar{\tau}_1 - T_1).$$

5. Limiting case $T_2 = \infty$.

Letting T_2 and β tend to infinity under the condition $\beta - 1 = BT_2$, where $B = \text{const}$, we obtain

$$F_\tau(t) = \frac{B^\alpha}{\Gamma(\alpha)} \int_{T_1}^{t} (\tau - T_1)^{\alpha-1} e^{-B(\tau - T_1)}\, d\tau,$$

whence

$$\bar{\tau} = T_1 + \alpha/B, \qquad \hat{\tau} = T_1 + (\alpha - 1)/B, \qquad D(\tau) = \alpha/B^2.$$

Using relation (5) to determine the estimates t_1, t_2, we obtain the validity of formulas (6). If the values $t_1, \hat{\tau}, t_2$ are given, then the distribution parameters are obtained in the form

$$\alpha = D(\tau)/(\bar{\tau} - \hat{\tau})^2, \qquad B = 1/(\bar{\tau} - \hat{\tau}), \qquad T_1 = \bar{\tau} - D(\tau)/(\bar{\tau} - \hat{\tau}).$$

The parameters are obtained just as simply if the sets $T_1, \hat{\tau}, t_2$ or $T_1, t_1, \hat{\tau}$ are given. In the case where $T_1, \hat{\tau}$ and the quantile T_q, where $q = 2/3$, are given, then to determine B one may use one of the iterative processes

$$B_{k+1} = \frac{2}{3} \frac{B_k}{\varphi(B_k)} \quad \text{or} \quad B_{k+1} = B_k - \frac{1}{A(B_k)} (\varphi(B_k) - {}^2/_3),$$

in which

$$\varphi(B) = \frac{B^\alpha}{\Gamma(\alpha)} \int_{T_1}^{T_q} (t - T_1)^{\alpha-1} e^{-B(t - T_1)}\, dt, \qquad \alpha = (\hat{\tau} - T_1)B + 1,$$

$A(B)$ is some positive function such that, in a neighborhood of the sought solution B_0, the following holds:

$$0 < \frac{d}{dB} \left[\frac{\varphi(B) - {}^2/_3}{A(B)} \right] < 2.$$

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Note: Figure translations are in progress. See original paper for figures.