ON THE UNSOLVABILITY OF THE WORD IDENTITY PROBLEM FOR ONE CLASS OF SEMIGROUPS WITH A SOLVABLE ISOMORPHISM PROBLEM

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Abstract

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ON THE UNSOLVABILITY OF THE WORD IDENTITY PROBLEM FOR ONE CLASS OF SEMIGROUPS WITH A SOLVABLE ISOMORPHISM PROBLEM

(Presented by Academician P. S. Novikov, February 4, 1966)

In the paper ([2]) the isomorphism problem was solved for the class K_1 of semigroups defined by systems of defining relations such that none of the defining words occurs in any one of the defining words distinct from it. The question of the solvability of the identity problem for these semigroups remained open, although for a certain subclass of the class of semigroups K_1 the identity problem was solved in ([1]).

Theorem 1. *The class K_1 contains a semigroup $\mathfrak{A}$ whose word identity problem has the highest recursively enumerable degree of unsolvability.**

The semigroup $\mathfrak{A}$ is defined by 15 generating elements: $a, b, c, d, e, f, x, y, p, r, s, q, m, n, v$, and 13 defining relations:

$$ad = da, \quad bc = cb, \quad bd = db, \quad ac = ca = vn, \quad aef = mn, \quad ccm = cv,$$

$$fc = rs, \quad sa = qf, \quad erq = ce, \quad fd = xy, \quad yb = pf, \quad exp = de.$$

The semigroup $\mathfrak{A}$ contains, as a subsemigroup, the semigroup $\mathfrak{B}$ of G. S. Tseitin ([6]), defined by generating elements a, b, c, d, e and defining relations:

$$ac = ca, \quad ad = da, \quad bc = cb, \quad bd = db,$$

$$eca = ce, \quad edb = de, \quad cca = ccae.$$

It is easy to verify that the mapping φ of the semigroup $\mathfrak{B}$ into the semigroup $\mathfrak{A}$, which sends

$$a \to a, \qquad b \to b, \qquad c \to c, \qquad d \to d, \qquad e \to ef,$$

is a homomorphism. The proof that φ is a monomorphism is omitted because of its cumbersome nature.

From the works of G. S. Tseitin ([6]) and P. S. Novikov ([5]) it is clear that the word identity problem of the semigroup $\mathfrak{B}$, and consequently also of the semigroup $\mathfrak{A}$, has the highest recursively enumerable degree of unsolvability.

Definition 1. By the *strictly restricted isomorphism problem* for a finitely defined semigroup $\mathfrak{A}$, given by generating elements $a_1, \dots, a_n$ and defining relations $A_i = B_i$ $(i = 1, \dots, k)$, we shall understand the following problem: to find an algorithm which, for any semigroup $\mathfrak{B}$ given by the same number n of generators and by a finite number of defining relations, makes it possible to determine whether the semigroup $\mathfrak{B}$ is isomorphic to the semigroup $\mathfrak{A}$, or to prove that no such algorithm exists.

Theorem 2. *Let the semigroup $\mathfrak{A}$ belong to the class K_1. Then the degree of unsolvability of the word identity problem of the semigroup $\mathfrak{A}$ is $\leq$ the degree of unsolvability of the strictly restricted isomorphism problem for $\mathfrak{A}$.*

* For the definition of the notion of the degree of unsolvability, see ([3]).

Theorem 2 easily follows from the theorem that the semigroup $\mathfrak{A}$ is not isomorphic to any of its proper factor-semigroups (the latter theorem is stated without proof in paper ([2]), Remark 3).

Theorem 3. *Let the semigroup $\mathfrak{A}$ belong to the class K_1. Then the degree of unsolvability of the strictly bounded isomorphism problem for the semigroup $\mathfrak{A}$ is $\leq$ the degree of unsolvability of the word identity problem for the semigroup $\mathfrak{A}$.*

In the proof of Theorem 3 the following lemma is used:

Lemma 1. *Let $\mathfrak{A}$ be a semigroup of the class K_1, given by generators $a_1, \dots, a_n$ and defining relations $A_i = B_i$ $(i = 1, \dots, k)$. Let $\mathfrak{B}$ be any semigroup isomorphic to the semigroup $\mathfrak{A}$ and given by generators $b_1, \dots, b_m$ and defining relations $C_j = D_j$ $(j = 1, \dots, s)$, where $C_j \neq D_j$ (for all j). Then any isomorphism φ of the semigroup $\mathfrak{B}$ onto the semigroup $\mathfrak{A}$ maps the set $\{b_k\}$ one-to-one onto $\{a_k\}$ $(k = 1, \dots, m)$; $\varphi^{-1}(A_i)$ and $\varphi^{-1}(B_i)$ are defining words of the semigroup $\mathfrak{B}$, and all defining words of the semigroup $\mathfrak{B}$ are nonempty. Moreover, if some defining word C_j or D_j of the semigroup $\mathfrak{B}$ does not contain another defining word of the semigroup $\mathfrak{B}$ as a proper subword, then the equality $\varphi(C_j) = \varphi(D_j)$ is a defining relation in the semigroup $\mathfrak{A}$.*

From Theorems 2 and 3 follows Theorem 4:

Theorem 4. *Let the semigroup $\mathfrak{A}$ belong to the class K_1. Then the degree of unsolvability of the strictly bounded isomorphism problem for the semigroup $\mathfrak{A}$ is equal to the degree of unsolvability of the word identity problem for the semigroup $\mathfrak{A}$.*

Theorem 5. *The class K_1 contains both an infinite number of pairwise nonisomorphic semigroups with solvable strictly bounded isomorphism problem and an infinite number of pairwise nonisomorphic semigroups, each of which is not isomorphic to any semigroup with a smaller number of generators and has the highest recursively enumerable degree of unsolvability of this problem.*

It is interesting to compare Theorem 5 with the result of A. A. Markov ([4]) that, for every semigroup with n generators, the problem of recognizing the isomorphism of this semigroup to any previously given semigroup with $n+4$ generators is unsolvable. In an unpublished work, G. S. Tseitin improved A. A. Markov' s result by replacing the number of generators $n+4$ by the number of generators $n+2$. Well-known examples of semigroups for which the strictly bounded isomorphism problem is unsolvable are, however, isomorphic to semigroups with a smaller number of generators.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.