CONDITIONS ON A SET, NECESSARY AND SUFFICIENT IN ORDER THAT EVERY CONTINUOUS FUNCTION ANALYTIC AT ITS INTERIOR POINTS ADMIT UNIFORM APPROXIMATION BY RATIONAL FUNCTIONS

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CONDITIONS ON A SET, NECESSARY AND SUFFICIENT IN ORDER THAT EVERY CONTINUOUS FUNCTION ANALYTIC AT ITS INTERIOR POINTS ADMIT UNIFORM APPROXIMATION BY RATIONAL FUNCTIONS

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The content of this note is a new scheme for approximating continuous functions of one complex variable, which has enabled us, in terms of analytic capacity, to formulate conditions on a set e in the complex plane that are necessary and sufficient for the coincidence of two algebras of functions: the algebra $A(e)$ of all functions continuous on e and analytic at its interior points, and the algebra $R(e)$ of uniform limits on e of rational functions with poles outside the set e. As a consequence of the result obtained we shall indicate one geometric criterion sufficient for the coincidence of the algebras $A(e)$ and $R(e)$.

Theorem 1. *If the set e is closed, bounded, and its interior boundary is empty, i.e. its entire boundary coincides with the union of the boundaries of the components of its complement, then $A(e) = R(e)$.*

Examples due to E. P. Dolzhenko ([1]) are known of sets with nonempty interior boundary for which $A(e) \neq R(e)$.

Denote by $C(e, m)$ the set of all functions each of which is continuous in the whole plane, analytic outside some closed subset of the set e, bounded in modulus by the constant m, and equal to zero at infinity; $\alpha(e) = \sup_{f \in C(e,1)} \lim_{z\to\infty} |zf(z)|$ is the analytic C-capacity of the set e; e^0 is the set of interior points of the set e; Ce is the complement of the set e; $K(z, \delta)$ is the closed disk with center at the point z and diameter δ.

Theorem 2. *Let e be a closed bounded set. For the coincidence of the algebras $A(e)$ and $R(e)$ it is necessary and sufficient that, for every open set g, the equality*

$$\alpha(Ce \cap g) = \alpha(Ce^0 \cap g)$$

hold.

This formulation as a conjecture was proposed by S. N. Mergelyan. The notion of C-capacity was introduced by E. P. Dolzhenko. For the case of a set e containing no interior points, it was shown earlier ([2]) that, for the coincidence of the algebras $A(e)$ and $R(e)$, it is necessary and sufficient that for every disk $K(z,\delta)$ the equality $\gamma(Ce \cap K(z,\delta)) = \gamma(K(z,\delta)) = \delta/2$ hold, where $\gamma(e)$ is the analytic capacity of the set e. This result agrees with Theorem 2, since for every open set e one has $\alpha(e) = \gamma(e)$.

Theorem 3. *Let e be a closed bounded set. In order that $A(e) = R(e)$, it is necessary and sufficient that, for every sequence of disks $K(z,\delta_n)$ $(n = 1, 2, \ldots)$ nested one in another with common center, there exist a constant $r \geq 1$ such that*

$$\lim_{n\to\infty} \frac{\alpha(Ce^0 \cap K(z,\delta_n))}{\alpha(Ce \cap K(z, r\delta_n))} < \infty.$$

Theorem 2 gives the strongest necessary condition for equality of the algebras, while Theorem 3 is more effective as a sufficient condition for the coincidence of the algebras.

Theorem 4. *If, for every boundary point z of the closed bounded set e,*

$$\lim_{\delta\to 0} \frac{\alpha(Ce \cap K(z,\delta))}{\delta} > 0,$$

then $A(e) = R(e)$.

Theorem 4 follows easily from Theorem 3. Theorem 1 follows from Theorem 4, since for every point z belonging to the exterior boundary of the set e, for sufficiently small δ, $\alpha(Ce \cap K(z,\delta)) > \frac{1}{8}\delta$, because in the present case the set $Ce \cap K(z,\delta)$ contains an arc of diameter $\delta/2$.

Theorems 2 and 3 are obtained from Theorems 5 and 6.

Theorem 5. *If a function $f(z) \in R(e)$ is analytic on the complement of some bounded open set e', then $f(z) \in R(e \cup Ce')$, i.e. this function admits uniform approximation by rational fractions on the set $e \cup Ce'$.*

Theorem 6. *Let e be a closed bounded set; let $m > 0$ and $r \geq 1$ be constants such that, for all z and δ,*

$$\alpha(Ce^0 \cap K(z,\delta)) \leq m\alpha(Ce \cap K(z, r\delta)).$$

Then $A(e) = R(e)$.

Theorem 3 is derived from Theorems 5 and 6 as follows. If $A(e) \neq R(e)$, then, by Theorem 6, for all constants m and r there exists a disk $K(z_1,\delta_1)$ of arbitrarily small radius δ_1 such that

$$\alpha(Ce^0 \cap K(z_1,\delta_1)) > m\alpha(Ce \cap K(z_1, r\delta_1)).$$

By Theorem 5, $A(e \cap K(z_1, \delta_1))$ certainly does not coincide with $R(e \cap K(z_1, \delta_1))$. Continuing this reasoning, we obtain a sequence of disks $K(z_n, \delta_n)$ such that $\delta_n \leq 1/2^n$, $K(z_{n+1}, \delta_{n+1}) \cap K(z_n, \delta_n) \neq 0$ $(n = 1, 2, ...)$, and

$$\underline{\lim}_{n\to\infty} \frac{\alpha(Ce^0 \cap K(z_n, \delta_n))}{\alpha(Ce \cap K(z_n, r\delta_n))} \geq m.$$

From the monotonicity of $\alpha(e)$ it is then not difficult to obtain a system of disks with a common center possessing the same property. The necessity of the condition of Theorem 3 is easily obtained from the necessity of the condition of Theorem 2, which in turn is easily obtained from Theorem 5.

Let us construct a special sequence of partitions of unity, i.e. a system of functions $\{g_{k,n}\}$ with the properties: 1) for every n,

$$\sum_{k=1}^{\infty} g_{k,n} \equiv 1;$$

2) outside the disk $K(z_{k,n}, \delta_n) = K(z_{k,n}, 4/n)$, $g_{k,n}(z) = 0$; 3) for every n, the number of disks from $\{K(z_{k,n}, \delta_n)\}$ containing one and the same point z does not exceed 25; 4) for all z' and z'',

$$|g_{k,n}(z') - g_{k,n}(z'')| \leq n|z' - z''|.$$

Let us agree on the following notation: consider the expression

$$\varphi(z) = \int_e \frac{\partial f}{\partial \bar{\zeta}}(\zeta) g(\zeta, z)\, dS,$$

where $f(\zeta)$ is a continuous, but in general nondifferentiable, function;

$$g(\zeta, z) = g_1(\zeta) + \frac{g_2(\zeta)}{\zeta - z};$$

g_1 and g_2 satisfy a Lipschitz condition. The indicated expression will be understood as the function defined by the equality

$$\varphi(z) = \int_e \frac{\partial}{\partial \bar{\zeta}} (f(\zeta) g(\zeta, z))\, dS - \int_e f(\zeta) \frac{\partial g(\zeta, z)}{\partial \bar{\zeta}}\, dS =$$

$$= 2\pi i f(z) g_2(z) + \int_{\partial e} f(\zeta) g(\zeta, z)\, d\zeta - \int_e f(\zeta) \frac{\partial g(\zeta, z)}{\partial \bar{\zeta}}\, dS.$$

Lemma 1. *If $f(z)$ is continuous on the whole plane and analytic on e^0, and $g(z)$ is a real-valued function satisfying a Lipschitz condition with constant n and equal to zero outside the disk $K(z_0, \delta)$, then*

$$\varphi(z) = \int \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{g(\zeta)}{\zeta - z}\, dS$$

is continuous on the whole plane, analytic on $e^0 \cup CK(z_0, \delta)$*, equal to zero at infinity, and bounded in modulus by the constant* $2\pi\delta n\omega(\delta)$*, where* $\omega(\delta)$ *is the oscillation of the function* $f(z)$ *in the disk* $K(z_0, \delta)$*.*

Proof of Theorem 5. Let δ be the distance from the set e_f of singular points of the function $f(z)$ (points of nonanalyticity) to the boundary of the set e'. Fix n so large that $\delta > 4/n = \delta_n$. We shall denote by the sign $\sum'$ summation over all such k for which $K(z_{k,n}, \delta_n) \cap e_f$ is nonempty. Let q be the number of such values of the index k, and let $r(z)$ be a rational function approximating $f(z)$ on e with accuracy $\varepsilon/16q$; let $f^*(z)$ be some function continuous in the whole plane, coinciding in some neighborhood e_ε of the set e with $r(z)$ and differing from $f(z)$ everywhere by no more than $\varepsilon/8q$. Put

$$f_\varepsilon(z) = \frac{1}{\pi} {\sum}' \int \frac{\partial f^*(\zeta)}{\partial \bar\zeta} \, \frac{g_{k,n}(\zeta)}{\zeta - z} \, dS.$$

By Lemma 1 this function is analytic on $e \cup Ce'$, and

$$|f(z) - f_\varepsilon(z)| \le \frac{1}{\pi} {\sum}' \left| \int \frac{\partial (f(\zeta) - f_\varepsilon(\zeta))}{\partial \bar\zeta} \, \frac{g_{k,n}(\zeta)}{\zeta - z} \, dS \right| \le \frac{1}{\pi} {\sum}' 2\pi n \delta_n \, \frac{\varepsilon}{8q} = \varepsilon.$$

Since ε is arbitrarily small, approximating $f_\varepsilon(z)$ on the set $e \cup Ce'$ by a rational function, we obtain the assertion of Theorem 5.

Let e be an open set. Denote

$$\beta(e, z, f(z)) = \frac{1}{(2\pi i)\alpha(e)} \int_{\partial e} f(\zeta)(\zeta - z) \, d\zeta, \qquad \text{where } f(\zeta) \in C(e, m); \qquad \beta(e, z) =$$

$$= \sup_{f \in C(e,1)} |\beta(e, z, f(z))|; \qquad \beta(e) = \inf_z \beta(e, z); \qquad O(e) \text{ is a point such that}$$

$$\beta(e, O(e)) = \beta(e).$$

Lemma 2. *If* $f(z) \in A(e)$*, and* $g(z)$ *is a real function satisfying a Lipschitz condition with constant* n *and equal to zero outside the circle* $K(z_0, \delta)$*, then*

$$\left| \int \frac{\partial f(\zeta)}{\partial \bar\zeta} \, g(\zeta) \, dS \right| \le 2\pi n \delta \, \omega(\delta) \, \alpha(Ce^0 \cap K(z_0, \delta)),$$

where $\omega(\delta)$ *is the modulus of continuity of* $f(z)$*.*

The lemma follows from the fact that the expression being estimated coincides with the first coefficient of the Laurent series for the function

$$\varphi(z) = \frac{1}{2\pi i} \int \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{\partial \zeta}{\zeta - z} \, dS,$$

whose maximum can be estimated with the aid of Lemma 1.

Lemma 3. *If the set e satisfies the conditions of Theorem 6 and $f(z) \in A(e)$, then for all k, n and every point z the inequality holds*

$$\left| \int \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \, g_{k,n}(\zeta)(\zeta - z) \, dS \right| \leq \mu\omega(\delta_n)\alpha(Ce \cap K(z, \rho\delta_n))\beta[(Ce \cap K(z, \rho\delta_n)), z],$$

where $\omega(\delta)$ is the modulus of continuity of $f(z)$; $\mu > 0$ and $\rho > 1$ are constants depending only on m and r (see Theorem 6).

The lemma is proved in the same way as Lemma 5 of work ([3]). We note here only that one should use the formula

$$\int \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \, g_{k,n}(\zeta)(\zeta - z) \, dS = \sum_q \int \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \, g_{k,n}(\zeta) q_{q,p}(\zeta)(\zeta - z) \, dS.$$

For the proof of Theorem 6 we consider the following construction. Fix a number n. Represent the function $f(z) \in A(e)$ in the form

$$f(z) = \sum_k f_{k,n}, \qquad f_{k,n} = \frac{1}{\pi} \int \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{g_{k,n}(\zeta)}{\zeta - z} \, dS.$$

By Lemma 1, $f_{k,n}$ is analytic outside the set

$$e^0_{k,n} = Ce^0 \cap K(z_{k,n}, \delta_n) \subset e_{k,n} = Ce \cap K(z_{k,n}, \rho\delta_n)$$

and does not exceed in modulus the constant $m_1\omega(\delta_n)$ (constants are denoted by $m_1, m_2, \ldots$). Represent $f_{k,n}$ in the form

$$f_{k,n}(z) = \sum_{s=1}^{\infty} a^{k,n}_s (z - O(e_{k,n}))^{-s}.$$

Lemmas 2 and 3 give us estimates for $|a^{k,n}_1|$ and $|a^{k,n}_2|$, which make it possible to use Lemma 6 of paper ([3]). By virtue of this lemma there is a function $\varphi_{k,n}(z) \in C(e_{k,n}, m)$, where $m = 5m_2\omega(\delta_n)$, for which the first two coefficients of the expansion in powers of $(z-O(e_{k,n}))^{-1}$ coincide respectively with $a_1^{k,n}$ and $a_2^{k,n}$. Therefore

$$|f_{k,n}(z)-\varphi_{k,n}(z)| \leqslant m_3\omega(\delta)\min\left\{1, \frac{\delta_n^3}{|z-z_{k,n}|^3}\right\}.$$

It now follows easily from this that the function $\varphi_n = \sum_k \varphi_{k,n}(z)$ uniformly approximates the function $f(z)$ on the set e with accuracy up to $m_4\omega(\delta_n)$. By the definition of $C(e_{k,n}, m)$, the function $\varphi_{k,n}$ is analytic in a neighborhood of the set e. Consequently, φ is analytic in a neighborhood of the set e. The theorem is proved.

Remark. For the case in which the set e has a nonempty interior boundary, it follows from the theorems obtained here only that, if the interior boundary consists of a finite number of points, then $A(e) = R(e)$. If it were possible to prove the semiadditivity of analytic C-capacity:

$$a(e_1 \cup e_2) \leqslant \text{const}[a(e_1) + a(e_2)],$$

then $A(e)$ would coincide with $R(e)$ in the case when the analytic C-capacity of the interior boundary of the set e is equal to zero.

The approximation construction considered above also makes it possible to prove the following theorem.

Theorem 7. Let e be a closed bounded set; let $f(z)$ be a function continuous on e and analytic on $e_0 \subset e$. Then this function can be uniformly approximated on e, with arbitrary accuracy, by a function $f^*(z)$ having the following properties: $f^*(z)$ is continuous on the whole plane, analytic on e_0, and analytic in some neighborhood of the exterior boundary of the set e.

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CITED LITERATURE

[1] E. P. Dolzhenko, DAN, **143**, No. 4, 771 (1962).
[2] A. G. Vitushkin, DAN, **128**, No. 1, 17 (1959).
[3] A. G. Vitushkin, DAN, **171**, No. 5 (1966).

Note: Figure translations are in progress. See original paper for figures.