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ON THE NATURE OF THE DEPENDENCE OF SOLUTIONS OF CERTAIN HYPERBOLIC EQUATIONS ON THE INITIAL FUNCTIONS

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In this note we consider the problem of the nature of the dependence of solutions of certain hyperbolic equations at the vertex of the characteristic cone on the initial functions on the base of the characteristic cone, i.e., the problem of the existence of lacunae and weak lacunae.

The most general definition of a lacuna was given in the well-known paper of I. G. Petrovskii ([1]); the definition of a weak lacuna was given in ([3], [6]). For linear hyperbolic equations with constant coefficients—and only such equations will be considered here—weak lacunae may be defined as regions on the base of the characteristic cone (triangle) in which the fundamental solution of the Cauchy problem is a polynomial in t; moreover, a weak lacuna has order k if this polynomial has degree $k-1$. Ordinary lacunae are those regions on the base of the characteristic cone in which the fundamental solution of the Cauchy problem is identically equal to zero.

Consider a strictly hyperbolic equation of order m with two independent variables and with constant coefficients:

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u \equiv L_0\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u + L_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u = 0; \tag{1}$$

here $L_0(\partial/\partial t, \partial/\partial x)$ is a homogeneous differential operator of order m, i.e., an operator containing only derivatives of order m; $L_1(\partial/\partial t, \partial/\partial x)$ is a differential operator containing lower derivatives (i.e., derivatives of order not higher than $m-1$).

In the case of the homogeneous equation $L_0(\partial/\partial t, \partial/\partial x)u = 0$, each interval between two successive characteristics on the base of the characteristic triangle is a weak lacuna of order not higher than $m-1$, as is seen from the well-known formula for the fundamental solution of the Cauchy problem for this equation (see, for example, ([5]), p. 242). For equation (1) the following holds.

Theorem 1. In order that each interval between two successive characteristics on the base of the characteristic triangle of equation (1) be a weak lacuna, it is necessary and sufficient that the operator $L(\partial/\partial t, \partial/\partial x)$ can be represented as a product of first-order differential operators of a certain form, namely:

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) \equiv \prod_{i=1}^{m}\left[\frac{\partial}{\partial t} - \lambda_i\left(\frac{\partial}{\partial x} - \varkappa\right)\right], \tag{2}$$

where $\varkappa$ is some constant, and λ_i $(i = 1, 2, \dots, m)$ are the roots of the characteristic equation $L_0(\lambda, 1) = 0$. In this case the order of the weak lacunae cannot exceed $m - 1$.

The sufficiency follows from the fact that the fundamental solution of the Cauchy problem for equation (2) has the form $K(t, x) = e^{\varkappa x} K_0(t, x)$, where $K_0(t, x)$ is the fun-

fundamental solution of the Cauchy problem for the equation $L_0(\partial/\partial t,\ \partial/\partial x)u = 0$. This is easily verified by direct substitution.

The proof of necessity is carried out according to the following scheme. Suppose that all λ_i are nonzero. According to (2) (p. 720), the fundamental solution of the Cauchy problem for equation (1) has the form

$$K(t, x) = \sum_{i=1}^{m} f_i(t, x)(x + \lambda_i t)^{m-2} \operatorname{sign}(x + \lambda_i t), \tag{3}$$

where $f_i(t, x)$ are certain analytic (in a neighborhood of the origin) functions of both variables. Carrying out the computations according to the algorithm proposed in (2), we obtain

$$f_i(t, x) = \sum_{k=0}^{\infty} \frac{e^{\beta_i t} P_{ki}(t)}{(m - 2 + k)!}(x + \lambda_i t)^k, \quad i = 1, 2, \dots, m;$$

here β_i is a certain constant depending on the coefficients of the equation, and $P_{ki}(t)$ are polynomials in t of degree not exceeding k.

On the other hand, from the condition that all intervals between each two consecutive characteristics are weak lacunae, it is not hard to obtain that $f_i(t, x)$ are polynomials in t with coefficients depending on x, of degree not exceeding some degree (denote it by p). This makes it possible to prove that $P_{ki}(t)$ do not depend on t, i.e., are constants. Then, considering $f_i(t, x)$ as a function of t and expanding it in powers of $(x + \lambda_i t)$, we obtain

$$f_i(t, x) = \sum_{k=0}^{p} \frac{1}{k! \lambda_i^k} c_{ki} e^{\varkappa_i x}(x + \lambda_i t)^k, \quad i = 1, 2, \dots, m,$$

where $\varkappa_i$ and c_{ki} $(i = 1, 2, \dots, m)$ are certain constants, with the c_{0i} such that the expression

$$K_0(t, x) = \sum_{i=1}^{m} c_{0i}(x + \lambda_i t)^{m-2} \operatorname{sign}(x + \lambda_i t)$$

is a fundamental solution of the homogeneous equation corresponding to equation (1). Substituting the obtained value of $f_i(t, x)$ into (3), from the initial conditions for $K(t, x)$ we find: 1) $c_{ki} = 0$ for $k = 1, 2, \dots, p$; $i = 1, 2, \dots, m$; 2) $\varkappa_1 = \varkappa_2 = \dots = \varkappa_m = \varkappa$. Then the fundamental solution of the Cauchy problem takes the form $K(t, x) = e^{\varkappa x} K_0(t, x)$, and, consequently, for equation (1) the weak lacunae (if they exist) are of the same order as for the equation $L_0(\partial/\partial t, \partial/\partial x)u = 0$, i.e., not higher than $m - 1$. Requiring that $K(t, x) = e^{\varkappa x} K_0(t, x)$ satisfy equation (1), we obtain that equation (1) must have the form (2).

In the case when one of the λ_i is equal to zero, the same proof goes through with only minor changes.

In paper (6) it was proved that for the equation

$$L_0\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u - cu = 0, \tag{4}$$

where $c \neq 0$ is a constant, none of the intervals between the characteristics in the base of the characteristic triangle is either a lacuna or a weak lacuna. This result is generalized to the case of n variables (n odd) for the following equation with a factorized principal part:

$$\prod_{i=1}^{m} \left(\frac{\partial^2}{\partial t^2} - \lambda_i^2 \Delta\right) u - cu = 0; \tag{5}$$

here

$$\Delta \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2};$$

$c \neq 0$ is a constant, $\lambda_1 > \lambda_2 > \dots > \lambda_{2m}$; $\lambda_i \neq 0$;

$$\lambda_i = -\lambda_{2m-i+1} \quad (i = 1, 2, \dots, m).$$

In what follows we shall regard x as a vector:

$$x = (x_1, x_2, \dots, x_n).$$

It is known [3] that for the equation

$$\prod_{i=1}^{m}\left(\frac{\partial^2}{\partial t^2}-\lambda_i^2\Delta\right)u=0$$

for odd n, the interior of the sphere with center at the point $(0,x)$ and radius $\lambda_m t$, lying in the base of the characteristic cone with vertex (t,x), is a lacuna in the case $n>2m$, and in the case $n<2m$ is a weak lacuna of order $2m-n$; moreover, each of the regions lying between two concentric spheres with center at $(0,x)$ and radii $\lambda_{i_0}t$ and $\lambda_{i_0-1}t$ ($i_0\le m$) in the base of the characteristic cone with vertex at (t,x) is a weak lacuna of order $2m-2$.

Theorem 2. *For equation (5), none of the regions into which the surface of the characteristic cone divides the plane of initial data is either a lacuna or a weak lacuna.*

The proof is carried out as follows: the fundamental solution of the Cauchy problem for equation (5) is constructed by reducing (5) to the corresponding equation with two independent variables (see [4], p. 297), which in our case will have the form (4). The fundamental solution of the Cauchy problem for equation (4) was constructed in [6] by the method of successive approximations in a somewhat different form than in [2]. Using the fundamental solution of the Cauchy problem for equation (4), we obtain (see [4]) the fundamental solution of the Cauchy problem for equation (5) in the form

$$K(t,x)=\sum_{l=0}^{\infty}A_l c^l K_l(t,x), \tag{6}$$

where $A_l\neq 0$ are certain constants;

$$K_l(t,x)=\sum_{i=1}^{2m}\sum_{k=0}^{l}\frac{d_{ki}^{(l)}}{(N(l)-k)!}\,t^k\int_{|\omega|=1}(x\omega+\lambda_i t)^{N(l)-k}\operatorname{sign}(x\omega+\lambda_i t)\,d\omega \tag{7}$$

for all l for which $N(l)\ge l$. Here and everywhere below $\omega=(\omega_1,\omega_2,\dots,\omega_n)$ is a unit vector; $d\omega$ is the area element of the unit sphere; $N(l)=2(l+1)m-n-1$; $d_{ki}^{(l)}$ are certain constants, recurrence formulas for which were obtained in [6].

For those l for which $N(l)<l$, and this is possible only if $2m-n-1<0$ and $l<(n+1-2m)/(2m-1)$, $K_l(t,x)$ have the form

$$\begin{aligned}K_l(t,x)=&\sum_{i=1}^{2m}\sum_{k=0}^{N(l)}\frac{d_{ki}^{(l)}}{(N(l)-k)!}\,t^k\int_{|\omega|=1}(x\omega+\lambda_i t)^{N(l)-k}\operatorname{sign}(x\omega+\lambda_i t)\,d\omega\\&+\sum_{i=1}^{2m}\sum_{k=0}^{l-N(l)-1}d_{N(l)+1+k,i}^{(l)}\,t^{N(l)+1+k}\int_{|\omega|=1}\delta^{(k)}(x\omega+\lambda_i t)\,d\omega,\end{aligned} \tag{8}$$

here $\delta^{(k)}(\alpha)$ is the derivative of order k of the delta function. In the case $N(l)<0$, only the second sum remains, with $d_{ki}^{(l)}=0$ for $k<0$.

Inside the sphere of radius $\lambda_m t$ in the base of the characteristic cone, formula (7) can be written in the form

$$K_l(t,x) = \sum_{k=0}^{N(l)/2} \sum_{i=1}^{m} b_{2k,i}^{(l)} t^{2k} |x|^{N(l)-2k};$$

between the spheres of radii $\lambda_{i_0} t$ and $\lambda_{i_0-1} t$ ($i_0 \leq m$) in the base of the characteristic cone, formula (7) takes the form

$$K_l(t,x) = \sum_{k=0}^{N(l)/2} \sum_{i=1}^{i_0-1} b_{2k,i}^{(l)} t^{2k} |x|^{N(l)-2k} + \sum_{k=0}^{N(l)/2+(n-3)/2} \sum_{i=i_0}^{m} b_{2k+1,i}^{(l)} t^{2k+1} |x|^{N(l)-(2k+1)}.$$

In these last formulas

$$b_{si}^{(l)} = B_s^{(l)} \sum_{\substack{r=0\\ r\leq l}}^{s} d_{ri}^{(l)} \frac{\lambda_i^{s-r}}{(s-r)!},$$

and $B_s^{(l)}$ are constants depending on m, n, l, and s.

Let $2m-n-1 \geq 0$. Then $N(l) \geq l$ for all l. As is seen from the formulas written above, in the domains under consideration $K_l(t,x)$ with different values of the index l have no similar terms, since the sum of the degrees of t and $|x|$ in $K_l(t,x)$ is equal to $N(l)$ for any l. Further, analogously to how this was done in (6), it is proved that in each of the domains $K_l(t,x)$ under consideration is a polynomial in t of degree not lower than l, whence the assertion of the theorem follows. If $2m-n-1 < 0$, the sum of several first terms of the series (6), for which $N(l) < l$, in the domains under consideration turns into polynomials in t or into an identically zero function (this follows from formula (8)); consequently, in this case the same proof also goes through.

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Note: Figure translations are in progress. See original paper for figures.