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Abstract

Full Text

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*CYBERNETICS
AND CONTROL THEORY*

B. I. KORENBLYUM

ESTIMATE OF THE TYPE OF AN ENTIRE FUNCTION AND THE PROBLEM OF OPTIMAL SPEED OF RESPONSE

(Presented by Academician A. Yu. Ishlinskii, 18 XII 1965)

1°. In the present paper, for entire functions $f(z)$ of order 1 (of exponential type), having on the real axis $-\infty < x < \infty$ a prescribed rate of decrease as $|x| \rightarrow \infty$, a lower estimate of the type is established, exact up to an absolute constant (coefficient). With the help of this estimate, the time of optimal speed of response in one control problem is determined (to within an absolute coefficient).

2°. **Theorem 1.** Let $f(z)$ be an entire function of order 1 and type T , satisfying on the real axis the conditions

$$\sup_{-\infty < x < \infty} |f(x)| = A_0, \quad (1)$$

$$|f(x)| \leq A_n |x|^{-n} \quad (-\infty < x < \infty; n = 1, 2, \dots), \quad (2)$$

where $\{A_n\}_0^\infty$ is a given sequence of positive numbers, which may (except for A_0) tend to $+\infty$, and

$$\lim_{n \rightarrow \infty} A_n^{1/n} = \infty. \quad (3)$$

Then

$$T \geq \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{A_{n-1}^c}{A_n^c}, \quad (4)$$

where $\{A_n^c\}_0^\infty$ is the convex regularization of the sequence $\{A_n\}$ by means of logarithms (see (1), p. 24). Conversely, whatever the sequence $\{A_n\}_0^\infty$ subject

to the indicated conditions and such that the series in (4) converges, there exists an entire function $f(z)$ of order 1 satisfying (1), (2) and having type

$$T = \sum_{n=1}^{\infty} \frac{A_{n-1}^c}{A_n^c}. \quad (5)$$

Remark 1. $\{A_n^c\}$ can also be defined in the following way: among the sequences $\{B_n\}$ satisfying the conditions

$$B_0/B_1 \geq B_1/B_2 \geq B_2/B_3 \geq \dots, \quad 0 < B_n \leq A_n \quad (n = 0, 1, 2, \dots),$$

the sequence $\{A_n^c\}$ is maximal:

$$A_n^c = \max B_n \quad (n = 0, 1, 2, \dots).$$

Remark 2. It is easy to see that $A_0^c = A_0$.

Remark 3. The sum in (4) increases together with A_0 . Therefore estimate (4) remains valid if (1) is replaced by the inequality

$$\sup_{-\infty < x < \infty} |f(x)| \geq A_0. \quad (1')$$

3°. We proceed to the proof of Theorem 1. We may assume that at least one A_n ($n \geq 1$) is finite, since for $A_n = +\infty$ ($n = 1, 2, \dots$) inequality (4) is trivial: $T \geq 0$. Therefore

$$f(x) \rightarrow 0 \quad (|x| \rightarrow \infty), \quad \max_{-\infty < x < \infty} |f(x)| = |f(x_0)| = A_0.$$

Introduce the function

$$k(x) = \min\{A_0, A_1/|x|, A_2/|x|^2, \dots\} \quad (-\infty < x < \infty). \quad (6)$$

Obviously,

$$|f(x)| \leq k(x) \quad (-\infty < x < \infty), \quad (7)$$

$$|f(x_0)| = k(x_0) = A_0. \quad (8)$$

Since the function $\log |f(x + iy)| - T|y|$ is bounded above and subharmonic in each of the half-planes $\pm y > 0$, using (7), the Poisson integral, and the properties of the harmonic majorant, we easily find

$$\log |f(x + iy)| \leq \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log k(t)}{(t-x)^2 + y^2} dt + T|y| \quad (y \neq 0).$$

For $x = x_0$ we obtain, by virtue of (8):

$$\log |f(x_0 + iy)| - \log |f(x_0)| \leq \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log k(t) - \log A_0}{(t-x_0)^2 + y^2} dt + T|y| \quad (y \neq 0),$$

$$\lim_{|y| \rightarrow 0} \frac{\log |f(x_0 + iy)| - \log |f(x_0)|}{|y|} \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log k(t) - \log A_0}{(t-x_0)^2} dt + T. \quad (9)$$

(The passage to the limit under the integral sign is possible, since the integrand changes monotonically as $|y| \downarrow 0$.)

Since $\log |f(z)|$ is harmonic in a neighborhood of the point $z = x_0$, at this point there must exist the derivative $\frac{\partial}{\partial y} \log |f(z)|$. By virtue of (9), this is possible only if

$$T \geq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log A_0/k(t)}{(t-x_0)^2} dt.$$

Obviously, $k(t) = A_0$ ($-|x_0| \leq t \leq |x_0|$); therefore the last inequality can be rewritten as

$$T \geq \frac{1}{\pi} \int_{|x_0|}^{\infty} \left(\log \frac{A_0}{k(t)} \right) \left[\frac{1}{(t-|x_0|)^2} + \frac{1}{(t+|x_0|)^2} \right] dt.$$

Using the inequality

$$\frac{1}{(t-|x_0|)^2} + \frac{1}{(t+|x_0|)^2} \geq \frac{2}{t^2} \quad (t > |x_0|),$$

we find

$$T \geq \frac{2}{\pi} \int_0^{\infty} t^{-2} \log \frac{A_0}{k(t)} dt \quad (10)$$

(by virtue of (6) and (3), the integrand is equal to zero in some neighborhood of the point 0).

We now need the following

Lemma.

$$\int_0^\infty t^{-2} \log \frac{A_0}{k(t)} dt = \frac{A_0^c}{A_1^c} + \frac{A_1^c}{A_2^c} + \dots \quad (11)$$

To prove the lemma, consider the function

$$M(s) = \max_{n \geq 0} (ns - \log A_n/A_0) \quad (-\infty < s < \infty).$$

In view of (6) we have

$$M(s) = \log \frac{A_0}{k(e^s)}, \quad \int_0^\infty t^{-2} \log \frac{A_0}{k(t)} dt = \int_{-\infty}^\infty e^{-s} M(s) ds.$$

Using the convexity of the function $M(s)$, it is easy to show by integration by parts that

$$\int_{-\infty}^\infty e^{-s} M(s) ds = \int_{-\infty}^\infty e^{-s} M'(s) ds = \int_{-\infty}^\infty e^{-s} dM'(s). \quad (12)$$

On the other hand, it is known (see (1), p. 25) that the function $M'(s)$ is piecewise constant and is equal to

$$M'(s) = 0 \quad (s < \log A_1^c/A_0^c),$$

$$M'(s) = n \quad (\log A_n^c/A_{n-1}^c < s < \log A_{n+1}^c/A_n^c; n \geq 1).$$

It follows that the integral (12) is equal to the sum (11). The lemma is proved. From (10) and (11) follows (4).

To prove the second part of Theorem 1, suppose that the series (4) converges, and put $\delta_n = A_{n-1}/A_n^c$ ($n \geq 1$). Obviously, $\delta_1 \geq \delta_2 \geq \dots$. Consider the function

$$f(z) = A_0 \prod_{n=1}^\infty \frac{\sin \delta_n z}{\delta_n z}.$$

Obviously,

$$\sup_{-\infty < x < \infty} |f(x)| = f(0) = A_0; \quad |f(x)| \leq A_0/\delta_1 \delta_2 \dots \delta_n |x|^n = A_n^c/|x|^n \quad (n \geq 1).$$

On the other hand, $f(z)$ is an entire function of order one, and its type is equal to $\delta_1 + \delta_2 + \dots$. The theorem is proved.

4°. We now consider the one-dimensional problem of optimal speed* for the control $d^n x/dt^n = u(t)$: it is required to transfer the point of the phase space $(x, x', \dots, x^{(n-1)})$ from the position $(0, 0, \dots, 0)$ to the position $(1, 0, \dots, 0)$ in the minimal time interval $(0, T)$, subject to the following restrictions on the phase coordinates and the control parameter u :

$$|x^{(k)}(t)| \leq A_k \quad (k = 1, 2, \dots, n-1); \quad |u(t)| = |x^{(n)}(t)| \leq A_n$$

$$(0 \leq t \leq T).$$

Somewhat generalizing this formulation, we arrive at the following problem:

A. Let $\{A_n\}_1^\infty$ be a prescribed sequence of positive numbers, allowed to tend to $+\infty$, and suppose that condition (3) is fulfilled. Find the lower bound of the numbers $T > 0$ such that there exists a function $x(t)$ ($-\infty < t < \infty$) satisfying the conditions

a)

$$x(t) = 0 \quad (t \leq 0); \tag{13}$$

b)

$$x(t) = 1 \quad (t \geq T); \tag{14}$$

c)

$$|x^{(n)}(t)| \leq A_n \quad (-\infty < t < \infty; n \geq 1)**. \tag{15}$$

Theorem 2. In problem A the estimate is valid ($A_0 = 1$)

$$T \geq \frac{4}{3\pi} \sum_{n=1}^{\infty} \frac{A_{n-1}^c}{A_n^c}. \tag{16}$$

* We follow the terminology and general ideas of the monograph (2).

** If some $A_{n_0} = \infty$, then the corresponding inequality (15) is not taken into account, i.e. the existence and boundedness of the derivative $x^{(n_0)}(t)$ is not assumed in advance.

Conversely, whatever the sequence $\{A_n\}_1^\infty$ for which the series (16) converges may be, there exists a nondecreasing function $x(t)$ ($-\infty < t < \infty$), satisfying (13)–(15) and such that

$$T = \frac{A_0^c}{A_1^c} + \frac{A_1^c}{A_2^c} + 2 \left(\frac{A_1^c}{A_3^c} + \frac{A_2^c}{A_4^c} + \dots \right). \quad (17)$$

Proof. Let $x(t)$ satisfy (13)–(15). Consider the entire function of order 1 and type $T/2$:

$$f(z) = \int_{-T/2}^{T/2} x' \left(t + \frac{T}{2} \right) e^{itz} dt. \quad (18)$$

Obviously, $f(0) = 1$. Integrating (18) by parts and using (15), we obtain the estimates

$$|f(x)| \leq A_{nT}/|x|^{n-1} \quad (-\infty < x < \infty; n \geq 1).$$

Now consider the function

$$f_1(z) = \frac{\sin Tz}{Tz} f(z)$$

of type $3/2T$. Obviously,

$$f_1(0) = 1, \quad |f_1(x)| \leq A_n/|x|^n \quad (-\infty < x < \infty; n \geq 1).$$

Applying Theorem 1 and Remark 3, we find

$$\frac{3}{2}T \geq \frac{2}{\pi} \left(\frac{A_0^c}{A_1^c} + \frac{A_1^c}{A_2^c} + \dots \right),$$

where $A_0 = 1$. The estimate (16) is proved.

Now suppose that the series on the right-hand side of (16) converges. To construct the function $x(t)$, we shall use an idea of S. Mandelbrojt and T. Bang (see (1), p. 106). Denote

$$\delta_1 = A_0^c/A_1^c, \quad \delta_2 = A_1^c/A_2^c, \quad \delta_n = 2A_{n-1}^c/A_n^c \quad (n \geq 3)$$

and consider the sequence of functions

$$x_0(t) = \begin{cases} 0 & (t \leq 0), \\ 1 & (t > 0); \end{cases} \quad x_n(t) = \frac{1}{\delta_n} \int_{t-\delta_n}^t x_{n-1}(\tau) d\tau \quad (-\infty < t < \infty; n \geq 1).$$

Refining somewhat the arguments of S. Mandelbrojt and T. Bang, it is not difficult to show that the limiting function

$$x(t) = \lim_{n \rightarrow \infty} x_n(t)$$

satisfies (13)–(15) and (17). Theorem 2 is proved.

Kyiv Civil Engineering Institute

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References

1. S. Mandelbrojt, *Adjacent Series. Regularization of Sequences*, Applications, Moscow, 1955.
2. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.

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