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Abstract

Full Text

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ON THE VARIATIONAL EQUATION OF STEADY-STATE CREEP OF SHELLS

1. The system of equations of the theory of steady-state creep is written as follows:

$$\sigma_{ij,j} + F_i = 0; \tag{1,1}$$

$$\sigma_{ij} = \partial U / \partial \varepsilon_{ij}; \tag{1,2}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad v_i = \dot{u}_i \tag{1,3}$$

To these equations one must add the boundary conditions

$$\sigma_{ij}\nu_j = \overline{T}_i \qquad (x_i \in \Sigma_T); \tag{1,4}$$

$$v_i = \dot{u}_i = \overline{v}_i \qquad (x_i \in \Sigma_v). \tag{1,5}$$

Here, as usual, σ_{ij} is the stress tensor; U is the potential function; ε_{ij} is the strain-rate tensor; u_i is the displacement vector; v_i is the velocity vector; quantities prescribed on the surface are marked by bars above; the volume of the body is V, and the surface is $\Sigma = \Sigma_T + \Sigma_v$. The system of equations written down is completely analogous to the system of equations of the nonlinear theory of elasticity for small displacements; the entire difference consists in the fact that the velocity vector v_i is replaced by the displacement vector u_i. This analogy is violated if the derivatives of the displacements cannot be considered small. Here we restrict ourselves to consideration of the geometrically linear theory.

Equations (1,1)–(1,3) are the Euler equations for the functional

$$I = \int_v [\sigma_{ij}(\varepsilon_{ij} - \tfrac{1}{2}v_{i,j} - \tfrac{1}{2}v_{j,i}) - U(\varepsilon_{ij}) + F_{iv}i]\, dV +$$

$$+\int_{\Sigma_T} \overline{T}_i v_i \, d\Sigma + \int_{\Sigma_v} \sigma_{ij}\nu_j(v_i - \overline{v}_i) \, d\Sigma, \tag{1,6}$$

and relations (1,4) and (1,5) are natural boundary conditions ([1]). In this case the quantities σ_{ij}, ε_{ij}, and v_i are regarded as independent functions.

By taking part of the natural conditions of the variational problem for functional (1,6) as preliminary conditions, one can obtain various variational principles for functionals containing a smaller number of independent functions, namely:

A. Reissner's principle ([2]). Equation (1,2) is satisfied; hence ε_{ij} are expressed uniquely through σ_{ij}, and consequently $\sigma_{ij}\varepsilon_{ij} - U(\varepsilon_{ij}) = \Phi(\sigma_{ij})$. Taking this relation into account, as well as the symmetry of the tensor σ_{ij}, we obtain

$$I = I_{\sigma v} = \int_v [-\sigma_{ij}v_{i,j} + \Phi(\sigma_{ij}) + F_{iv}i] \, dV +$$

$$+\int_{\Sigma_T} \overline{T}_i v_i \, d\Sigma + \int_{\Sigma_v} \sigma_{ij}\nu_j(v_i - \overline{v}_i) \, d\Sigma. \tag{1,7}$$

B. Lagrange principle. If (1.3) and (1.5) are satisfied, then

$$I = I_v = \int_v [-U(e_{ij}) + F_i v_i] \, dV + \int_{\Sigma_T} \overline{T_i} v_i \, d\Sigma. \tag{1.8}$$

C. Castigliano principle. If (1.2), (1.1), and (1.4) are satisfied. Integrating by parts the first term in the volume integral of expression (1.7), we obtain:

$$I = I_\sigma = \int_v \Phi(\sigma_{ij}) \, dV - \int_{\Sigma_v} \sigma_{ij} v_j \overline{v_i} \, dV. \tag{1.9}$$

2. In applications of creep theory or of the nonlinear theory of elasticity to problems of a particular nature, one usually encounters certain kinematic restrictions on the possible deformation and the corresponding special features of the stress distribution. Following the same path of the involutory transformation of the variational problem for the functional (1.6), it sometimes proves possible to substantially reduce the number of functions subject to variation. In this way mixed variational equations are obtained, containing, as independent functional arguments, certain generalized forces and velocities. For the problem of axisymmetric creep of a circular cylindrical shell, such a variational equation was obtained independently in ([3]), where the actual shell was replaced by a two-layer model. The simplifications that arose were essentially connected with the circumstance that, in the case under consideration, the force T_{11} is known in advance, and the rate of change of curvature $\chi_{22} = 0$. As was noted in ([3], [4]), these conditions are also satisfied approximately for certain other problems of shell theory, in particular for long cylindrical shells (semimomentless theory). It is therefore of interest to obtain the variational equation of the technical theory of shells (the author' s term, see ([4])), following the general method. Such an approach also appears advantageous in the following respect. If the assumptions of the technical theory are satisfied only approximately, different methods of simplifying the equations may lead to results differing by small terms. Among equivalent variants, the one for which there exists an equivalent variational formulation will possess an undoubted advantage. In this case it is sufficient to make definite kinematic hypotheses; the corresponding simplifications in the static equations follow from them automatically.

Following mainly the notation of ([3]), set

$$\sigma_{11}^{\pm} = \frac{2\sigma_0}{\sqrt{3}}(\tau_1 \pm m_1), \quad \sigma_{22}^{\pm} = \frac{2\sigma_0}{\sqrt{3}}(\tau_2 \pm m_2), \quad \varepsilon_{11}^{\pm} = \varepsilon_0(\varepsilon_1 + \chi_1), \quad \varepsilon_{22}^{\pm} = \varepsilon_0\varepsilon_2.$$

We shall assume that a normal pressure q acts on the shell, and that the component of the velocity in the direction of the normal to the surface is v_n. The boundary conditions will be taken such that the functional reduces to an integral over the surface of the shell and contour integrals do not appear in the expression for the functional. The functional (1.7) takes the following form (up to a factor):

$$I = \int \left[-\frac{4}{\sqrt{3}}(\tau_1\varepsilon_1 + \tau_2\varepsilon_2 + m_1\chi_1) + \varphi\left(\frac{s^+}{\sigma_0}\right) + \varphi\left(\frac{s^-}{\sigma_0}\right) + \frac{qv_n}{\delta\sigma_0\varepsilon_0} \right] dS. \quad (2.1)$$

Here it is assumed that

$$\Phi(s) = \sigma_0\varepsilon_0\,\varphi\left(\frac{s}{\sigma_0}\right), \quad (2.2)$$

$$(s^{\pm})^2 = \frac{4}{3}\sigma_0^2[(\tau_1 \pm m_1)^2 + (\tau_2 \pm m_2)^2 - (\tau_1 \pm m_1)(\tau_2 \pm m_2)].$$

We also put $v = \varepsilon_0\psi'(s/\sigma_0)$, $\omega = v\sigma_0/s\varepsilon_0$.

Varying the functional (2.1) with respect to m_2 and τ_2, we obtain the conditions

$$\omega^+(\tau_2 + m_2 - \tfrac{1}{2}\tau_1 - \tfrac{1}{2}m_1) - \omega^-(\tau_2 - m_2 - \tfrac{1}{2}\tau_1 + \tfrac{1}{2}m_1) = 0, \quad (2.3)$$

$$\omega^+(\tau_2 + m_2 - \tfrac{1}{2}\tau_1 - \tfrac{1}{2}m_1) + \omega^-(\tau_2 - m_2 - \tfrac{1}{2}\tau_1 + \tfrac{1}{2}m_1) = \sqrt{2}\varepsilon_2. \quad (2.4)$$

We shall now take (2.3) and (2.4) as preliminary conditions of the variational problem and transform the functional (2.1). From (2.4) it follows that

$$\tau_2 \pm m_2 = \frac{1}{2}(\tau_1 \pm m_1) + \frac{\sqrt{3}}{2}\frac{\varepsilon_2}{\omega^\pm},$$

and (2.2) leads to the relation defining the functions $\omega^\pm$:

$$\frac{1}{\varepsilon_0^2} v^2(\omega^\pm) = \varepsilon^2 + (m \pm \tau)^2 (\omega^\pm)^2. \tag{2.5}$$

Here and below the indices on ε_2, m_1, and τ_1 are omitted. Setting [3]

$$\psi = \frac{1}{\varepsilon_0^2} \int \frac{dv^2}{\omega},$$

we represent the desired functional in the form

$$I = \iint \left\{ -\frac{4}{\sqrt{3}} m\chi - \frac{2}{\sqrt{3}}\tau\varepsilon + (m+\tau)^2\omega^+ + (m-\tau)^2\omega^- - \frac{1}{2}\psi(\omega^+) - \frac{1}{2}\psi(\omega^-) + qv_n/\sigma_0\varepsilon_0\delta \right\} dS. \tag{2.6}$$

The integral is extended over the surface S of the shell; the quantities χ, ε are expressed through the velocities of displacement of the points of the shell; these velocities, as well as the quantity m, are varied, whereas τ is prescribed.

3. Let us apply the results obtained to semi-momentless cylindrical shells. Direct axis 2 along the generator, axis 1 in the perpendicular direction, and axis 3 along the normal. For simplicity we shall proceed from Vlasov' s kinematic hypotheses [5], although the derivation can also be carried out under less restrictive assumptions (see [6]). In consequence of these hypotheses

$$v_{2,1} + v_{1,2} = 0, \qquad v_{1,1} + \frac{1}{\rho} v_3 = 0.$$

The first of these relations can be satisfied by taking $v_1 = -h^2\dot{\alpha}_{,1}$, $v_2 = h^2\dot{\alpha}_{,2}$. Then $v_3 = \rho h^2 \dot{\alpha}_{,11}$, $\varepsilon = h^2\dot{\alpha}_{,22}$,

$$\chi = h^3 \left[(\rho\dot{\alpha}_{,11})_{,11} + \left(\frac{\dot{\alpha}_{,1}}{\rho}\right)_{,1} \right] = \Omega(\dot{\alpha}).$$

Here dots denote differentiation with respect to the dimensionless time $\varepsilon_0 t$.

Now

$$I = \iint \left\{ -\frac{4h^3}{\sqrt{3}} \left(\rho \dot{\alpha}_{,11} m_{,11} - \frac{1}{\rho} \dot{\alpha}_{,1} m_{,1} \right) - \frac{2}{\sqrt{3}} \tau \dot{\alpha}_{,22} + (m+\tau)^2 \omega^+ \right.$$

$$\left. + (m-\tau)^2 \omega^- - \frac{1}{2} \psi(\omega^+) - \frac{1}{2} \psi(\omega^-) + \frac{q \rho h^2}{\sigma_0 \delta} \dot{\alpha}_{,11} \right\} dS. \qquad (3.1)$$

Equating the variation of the functional (3.1) to zero, we obtain the equations

$$-\frac{4}{\sqrt{3}} \Omega(\varepsilon) - h^2 \left[(m+\tau)\omega^+ + (m-\tau)\omega^- \right]_{,22} = 0,$$

$$-\frac{4}{\sqrt{3}} \Omega(m) - h^2 \left[\left(\frac{1}{\omega^+} + \frac{1}{\omega^-} \right) \varepsilon \right]_{,22} - \frac{h^2}{\sigma_0 \delta} (q\rho)_{,11} = 0 \qquad (3.2)$$

and the natural boundary condition at the ends of the shell

$$\left\{ \left[(m+\tau)\omega^+ + (m-\tau)\omega^- \right] \delta\alpha_{,2} - \left[(m+\tau)\omega^+ + (m-\tau)\omega^- \right]_{,2} \delta\alpha \right\}_0^l = 0. \quad (3.3)$$

Condition (3.3) will be satisfied, in particular, if the shell is supported at the ends. Indeed, if $\alpha(0) = \alpha(l) = 0$, then $v_3(0) = v_3(l) = 0$; if $m(0) = m(l) = 0$, then $\omega^+(0) = \omega^-(0)$, and the first bracket vanishes.

expressions (3.3). Equations (3.2) (for $\tau = 0$) were obtained in [4].

The variational equation (3.1) can also be applied to the solution of shell-stability problems. For this it suffices to put in this equation $q = T_{ij} k_{ij}$, where k_{ij} is the change of curvature. In this way one can consider the problem of stability of a shell subjected to external pressure; in this case

$$q = \frac{4\sigma_0}{\sqrt{3}} \frac{\delta}{h} \tau \Omega(a).$$

The application of numerical methods to the solution of the corresponding variational problem is, in principle, carried out in the same way as in analogous one-dimensional problems [3, 7], but the amount of computation correspondingly increases. It is not difficult to determine the critical time for a very long tube, when the influence of the end restraints may be neglected; this problem was considered in [8]. Choosing the form of loss of stability to be the same as in the case of an elastic shell, one can determine the critical time also with allowance for finite length. Beginning from a certain instant of time, when the deflection becomes sufficiently large, the quantity ω depends to a greater degree on ε than on m and τ; therefore the simplifications made in [7] are possible. As a result we arrive at the following differential equation for the deflection coefficient a:

$$a = \frac{k_1}{\tau}\,\dot{a}^{1/n} - k_2\tau\dot{a}^{-1/n}. \tag{3.4}$$

This equation differs from that obtained in [7] for the axisymmetric problem by the absence of a free term. The difference is that in our case stability is considered with respect to perturbations that are orthogonal to the principal form of deflection due to the action of the external load, whereas in the axisymmetric case loss of stability is understood as the attainment of an infinitely large value of the deflection in the fundamental motion. Nevertheless, equation (3.4) makes it possible to determine a critical time proportional to τ^{-n}, which depends only weakly on the initial value $a_0 = a(0)$ and turns out to be finite for $a_0 = 0$. The reason for this situation is that the assumption of the smallness of m in comparison with ε is not fulfilled in the initial phase of motion. It is of interest to determine by direct calculations what actual error is introduced by the indicated simplification, which makes it possible to find, at least, a lower bound for the critical time.

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Note: Figure translations are in progress. See original paper for figures.