CONSTRUCTIVE REPRESENTATION OF NULL CLASSES OF DIFFERENTIABLE FUNCTIONS OF MANY VARIABLES

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Abstract

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CONSTRUCTIVE REPRESENTATION OF NULL CLASSES OF DIFFERENTIABLE FUNCTIONS OF MANY VARIABLES

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In our joint paper with J. Lions and P. I. Lizorkin (^{11})* and in the paper of M. Taibleson (^{16}), it is shown from various points of view that one can define null classes $B^0_{p\theta}$ $(1 \leq \theta < \infty)$ and $H^0_p = B^0_{p\infty}$ of generalized functions f in such a way that the Bessel-Macdonald operator (^{15})

$$J_r^\wedge f = (1+|x|^2)^{-r/2}\hat{f} = G_r^\wedge * f, \tag{1}$$

$$G_r(|x|) = \frac{K_{(n-r)/2}(|x|)}{2^{(r-2)/2}\Gamma(r/2)|x|^{(n-r)/2}} \qquad \left(r > 0, \quad |x|^2 = \sum_{j=1}^{n} x_j^2\right),$$

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \xi^{-\nu-1} e^{-\xi - z^2/4\xi}\, d\xi$$

transforms them respectively onto the known functional classes $B^0_{p\theta}$ and H^r_p $(J^0_{rB_{p\theta}} = B^r_{p\theta})$, defined as follows. Let R be the n-dimensional space of points $x = (x_1, \dots, x_n)$ with real coordinates; $1 \leq p \leq \infty$; L_p the space of functions f with finite norm

$$\|f\|_p = \left(\int |f|^p\, dx\right)^{1/p} < \infty \qquad \left(\int = \int_R\right),$$

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h),$$

$r > 0,\ r = \bar{r} + \alpha$, where $\bar{r}$ is an integer and $0 < \alpha \leq 1$. By definition, a function $f \in H^r_p$ if the finite norm

$$\|f\|_{H^r_p} = \|f\|_p + M_f < \infty, \tag{2}$$

is meaningful for it, where $M = M_f$ is the least constant for which the inequalities

$$\sup_{|h|\le t} \|\Delta_h^2 f(x)\|_p = \omega^2(f,t)_p \le M|h|^\alpha$$

hold for all partial derivatives $f^{(\bar r)}$ of f of order $\bar r$. By definition, a function f belongs to the class $B^r_{p\theta}$ (O. P. Besov) if the finite norm

$$\|f\|_{B^r_{p\theta}} = \|f\|_p + \sum \left(\int_0^\infty \frac{\omega^2\left(f^{(\bar r)}, t\right)_p^\theta}{t^{1+\alpha\theta}}\, dt \right)^{1/\theta}, \tag{3}$$

is meaningful for f, where the sum is extended over all partial derivatives $f^{(\bar r)}$ of f of order $\bar r$.

The class H^r_p is equivalent (see (^{11})) to the class $H_p^{(r,\dots,r)}$ (see (^{12})), just as the class $B^{(r)}_{p\theta}$ is equivalent to the class $B^{r,\dots,r}_{p\theta}$ (see (^{3,4}) and (^{13})).

In the works (^{11,16}) it was also established that the operation $J_r f$ (which has meaning for any real r) maps the class $B^\rho_{p\theta}$ ($\rho > 0$) onto the class $B^{r+\rho}_{p\theta}$ ($r+\rho > 0$) one-to-one and isomorphically in the sense of the metrics of these classes.

M. Taibleson obtained these results by considering the functions under study as traces of solutions of the heat equation; P. I. Lizorkin, by considering them as traces of solutions of the metaharmonic equation; Lions approached this question on the basis of interpolation of the classes $B^r_{p\theta}$ and $B^{-r}_{p\theta} = (B^r_{p'\theta'})^*$ ($1/p + 1/p' = 1,\ 1/\theta + 1/\theta' = 1$); and the author of the present paper, from the point of view of approximating functions of the indicated classes by functions of exponential type.

The results reported below are a development of the indicated investigations of ours (in the spirit of approximation theory) and of the recent results of M. D. Ramazanov (^{14}) and P. I. Lizorkin (^{9}). As in the cited papers, we proceed from the well-known class S (L. Schwartz) (see ($^{15,\,5-7}$)) of basic functions φ. A function $\varphi \in S$ is complex-valued, infinitely differentiable on R, and such that for any positive number l and nonnegative integral vector $\mathbf{k} = (k_1, \dots, k_n)$ the supremum

$$\sup_{x\in R} (1 + |x|^l)|\varphi^{(\mathbf{k})}(x)| = \varkappa(l, \mathbf{k}; \varphi) < \infty$$

is finite. Convergence is defined in S: if $\varphi_m, \varphi \in S$ and $\varkappa(l, \mathbf{k}; \varphi_m - \varphi) \to 0$ ($m \to \infty$) for all pairs $(l, \mathbf{k})$, then one writes $\varphi_m \to \varphi$.

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* This paper arose in connection with conversations among the three authors in Moscow in May 1963.

On S a functional (a generalized function f) (f, φ) is defined: linear, i.e. if c_1, c_2 are complex numbers and $\varphi_1, \varphi_2 \in S$, then

$$(f, c_1\varphi_1 + c_2\varphi_2) = c_1(f, \varphi_1) + c_2(f, \varphi_2), \tag{4}$$

and continuous: $(f, \varphi_m) \to (f, \varphi)$, if $\varphi_m \to \varphi$.

The generalized functions constitute the space S'. For functions $f \in S'$, the direct and inverse Fourier transforms $\tilde{f}, \hat{f}$ are defined by means of the equalities $(\tilde{f}, \varphi) = (f, \tilde{\varphi})$, $(\hat{f}, \varphi) = (f, \hat{\varphi})$, where $\tilde{\varphi}, \hat{\varphi}$ are the corresponding Fourier transforms of φ.

We have followed the manner of V. S. Vladimirov (^{6}) in defining the functions $f \in S'$. With another definition (see (^{7})), in the right-hand side of (4) instead of the numbers c_1, c_2 there stand their conjugates, and then $\tilde{f}$ has to be defined with the help of the equality $(\tilde{f}, \varphi) = (f, \varphi)$. If ψ is an infinitely differentiable function on R with positive growth of its derivatives and $f \in S'$, then ψf is defined as the functional $(\psi f, \varphi) = (f, \psi\varphi)$, belonging to S'. Therefore the operation (1) has meaning for any real r. It maps S' onto S' one-to-one and continuously in the sense of (S).

Let $1 \leq p \leq \infty$. We define the notion of a generalized function regular in the sense of L_p. A function $f \in S'$ is called regular in the sense of L_p (belonging to S'_p) if there exists a $\rho > 0$ such that $J_\rho f \in L_p$. Then already for $\rho' > \rho$ automatically $J_{\rho'} f \in L_p$. We define the convolution of an arbitrary summable on R function $K(x) \in L$ with $f \in S'_p$ by means of the equality

$$K * f = J_{-\rho}(K * J_\rho f). \tag{5}$$

This definition does not depend on ρ and is such that equality (5) remains valid for $f \in S'_p$ for any real ρ.

Denote by $\mathfrak{M}_\nu L_p$ $(\nu > 0)$ the class of functions $g_\nu \in L_p$ of exponential type of degree ν. If $g_\nu \subset \mathfrak{M}_\nu L_p$ and $\rho < 0$, then the inequality

$$\|J_\rho g_\nu\|_p \leq \varkappa_p \nu^{-\rho} \|g_\nu\|_p \tag{6}$$

(an analogue of Bernstein' s inequality) (see (^{8,11})), where $\varkappa_p$ depends on p, holds. We also prove that inequality (5) remains valid also for $\rho > 0$,

if the Fourier transform $\tilde{g}$ (in general belonging to S') has support outside the cube $\Delta_\nu = \{|x_j| \leq \nu;\ j = 1, \ldots, n\}$ (an analogue of Favard' s inequality (^{10})).

We also introduce the kernel (an analogue of the periodic Vallée-Poussin kernel)

$$V_N(t) = \frac{1}{N^n} \int_{\Omega_N} \prod_{j=1}^{n} \frac{\sin \lambda_j t_j}{t_j}\, d\lambda, \qquad \Omega_N = \{N < \lambda_j < 2N;\ j = 1, \ldots, n\}.$$

This is a special case of kernels that were studied by N. I. Akhiezer and B. M. Levitan (see (^{2}) and (^{1}), § 90). The following properties of V_N are important: $V_N \in \mathfrak{M}_{2N} L_p$; $\tilde{V}_N$ has support equal to 1 on Δ_N; $\|V_N\|_L \leq M$, where M does not depend on $N \geq 1$. For every function f regular in the sense of L_p (generalized) ($f \in S'_p$), one can define the convolution

$$\sigma_N(f, x) = \left(\frac{2}{\pi}\right)^{n/2} (V_N * f),$$

which is an ordinary function belonging to $\mathfrak{M}_{2N} L_p$. If $f \in L_p$, then, on the basis of the indicated properties of V_N, we have $\|\sigma_N(f, \mathbf{x}) - f(\mathbf{x})\|_p \to 0$ $(N \to \infty)$. But if $f \in S'_p$, then we prove that, in any case, weak convergence takes place,

$$\sigma_N(f, \mathbf{x}) \to f(\mathbf{x}) \quad (S),$$

i.e. $(\sigma_N(f), \varphi) \to (f, \varphi)$ for all $\varphi \in S$.

We define the class $B^0_{p\theta}$ $(1 \leq p \leq \infty,\ 1 \leq \theta \leq \infty,\ B^0_{p\infty} = H^0_p)$ as follows: $f \in B^0_{p\theta}$ if $f \in S'_p$ and if

$$\|f\|_{B^0_{p\theta}} = \|\sigma_{2^0}(f)\|_p + \left\{\sum_1^{\infty} \|\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)\|_p^\theta \right\}^{1/\theta} < \infty. \tag{7}$$

Note that, by what was said above, for every function $f \in S'_p$ the series

$$f = \sigma_{2^0}(f) + \sum_1^{\infty} [\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)] \quad (S), \tag{8}$$

converging weakly to f, is meaningful. To it one may apply the operation J_ρ term by term for any real ρ:

$$J_\rho f = J_\rho \sigma_{2^0}(f) + \sum_{k=1}^{\infty} J_\rho [\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f)] =$$

$$= \sigma_{2^0}(J_\rho f) + \sum_{k=1}^{\infty} [\sigma_{2^k}(J_\rho f) - \sigma_{2^{k-1}}(J_\rho f)] \quad (S). \tag{9}$$

Since for $f \in S'_p$ the function $\sigma_{2^k}(f) - \sigma_{2^{k-1}}(f) \in \mathfrak{M}_{2^{k+1}} L_p$, and its support lies outside Δ_{2^k}, inequality (6) is applicable to it with $\nu = 2^{k+1}$ for any real ρ. Hence it follows that norm (7) is equivalent to the norm

$$\|F\|_{B^0_{p\theta}} = \|\sigma_{2^0}(F)\|_p + \left\{ \sum_{k=1}^{\infty} 2^{k\theta\rho} \|\sigma_{2^k}(F) - \sigma_{2^{k-1}}(F)\|_p^\theta \right\}^{1/\theta}. \tag{10}$$

One can show that this norm is equivalent to norm (2) for $\theta = \infty$ and to (3) for $1 \le \theta < \infty$. For this it is simplest to obtain an equivalence of (10) with another norm (see ([3,4])), expressed in the language of best approximations, for which it is already known that it is equivalent respectively to norms (2) and (3).

It follows from what has been said that the operation J_ρ, for $\rho > 0$, realizes an isomorphism of the classes

$$J_\rho(B^0_{p\theta}) = B^\rho_{p\theta} \tag{11}$$

in the sense of the norms of these classes. But equality (11) also defines the classes $B^0_{p\theta}$ for negative ρ. As was already noted above, Lions dealt with these classes.

It follows from what has been said that if ρ is a real number, $1 \le p \le \infty$ and $1 \le \theta \le \infty$, then $F \in B^\rho_{p\theta}$ if and only if $F \in S'_p$ and the norm (10) is finite. In this case the isomorphism (11) holds.

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Note: Figure translations are in progress. See original paper for figures.

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