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THEORY OF ELASTICITY

1966

SovietRxiv

Abstract

Full Text

UDC 539.37

THEORY OF ELASTICITY

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Let a thin plate have a load-free hole and be in a homogeneous field of tensile stresses. In this case, near the holes, generally speaking, regions of compressive stresses arise. Let us imagine that the plate is made of a polymeric material or of a polymeric material reinforced with glass fiber (fiberglass). A characteristic property of viscous materials, including polymers, is that plates made of such materials, upon application of even small compressive longitudinal loads, begin to buckle. Fiberglass materials possess an analogous property, since the stability of glass filaments under compressive longitudinal loads is very low. Simultaneously with the local loss of stability, relaxation of the compressive stresses takes place in the buckled region. Both these processes lead to a redistribution of stresses near the hole. Ultimately, at times comparable with the characteristic time of loss of stability ([1]), a certain stationary buckling zone is formed near the hole, and a time-independent distribution of stresses is established in the plate. Similar considerations are valid in the case of longitudinal forces applied to a solid plate. In this connection the following problems arise: 1) determination of the limiting stress state for plates with a prescribed hole contour; 2) finding such shapes of holes that the strength of a plate with a hole would be as great as possible. These problems are considered below.

§ 1. Let the material of the plate under conditions of tension obey a linear law of viscoelasticity of general form. The problem of determining the limiting stress state for plates with a prescribed hole contour is posed as follows. It is required to find the stresses in the buckled and unbuckled regions of the plate and the boundary of separation of these regions itself, not known in advance. The limiting stress state in the buckled region, obviously, will be the same as for plates that do not sustain compressive stresses (membranes). The latter problem was studied in work ([2]). It is shown that in the buckled region of a membrane an identically zero stress state occurs if, on a known boundary of the buckled zone (which is part of the prescribed hole contour), the external loads are equal to zero. In this case, by a membrane one may also understand a plate with zero flexural rigidity ([3]). For the unbuckled region of the plate, the viscoelastic analogy will hold, since along the entire contour of this region the boundary conditions are imposed for the stresses. Thus, the limiting stress state in a viscoelastic plate (or in a fiberglass plate) will be the same as for a membrane, so that all the results of works ([2–4]) carry over entirely to the problem under consideration.

As examples illustrating the effect of buckling, we shall give ready-made solutions of two problems.

Let the hole in the plate be a slit $(-l, +l)$, located along the x-axis (x, y are Cartesian coordinates). The stresses at infinity are equal to $\sigma_x = \sigma_\infty$, $\sigma_y = \sigma_y^\infty$, $\tau_{xy} = 0$. The Muskhelishvili potentials–

the Muskhilishvili potentials $\Phi(z)$ and $\Psi(z)$ in the unbuckled region in this case are given by formulas (3):

for $\sigma_x^\infty < \sigma_y^\infty \leqslant 5\sigma_x^\infty$

$$\Phi(z) = \frac{(\sigma_x^\infty + \sigma_y^\infty)\zeta}{4\sqrt{\zeta^2 - 1}}, \qquad z = x + iy = l\zeta - Al\left[(\zeta^2 - 1)^{3/2} - \zeta^3 + \zeta\right],$$

$$\Psi(z) = -l\Phi'(z)\left\{\zeta + A\left[(\zeta^2 - 1)^{3/2} + \zeta^3 - \zeta\right]\right\}, \qquad A = 2\frac{\sigma_y^\infty - \sigma_x^\infty}{\sigma_y^\infty + 3\sigma_x^\infty};$$

for $\sigma_y^\infty > 5\sigma_x^\infty$

$$\Phi(z) = \frac{4(\sigma_x^\infty + \sigma_y^\infty)\zeta^2 + \sigma_y^\infty - 5\sigma_x^\infty}{16\zeta\sqrt{\zeta^2 - 1}},$$

$$\Psi(z) = -l\Phi'(z)\left[\zeta^3 + (\zeta^2 - 1)^{3/2}\right], \qquad z/l = \zeta^3 - (\zeta^2 - 1)^{3/2}. \tag{1}$$

The contour of the boundary of the buckled zone is specified by the equations:

for $\sigma_x^\infty < \sigma_y^\infty \leqslant 5\sigma_x^\infty$

$$x/l = At(t^2 - 1) + t, \qquad y/l = A(1 - t^2)^{3/2} \qquad (1 \geqslant t \geqslant -1);$$

for $\sigma_y^\infty > 5\sigma_x^\infty$

$$x^{2/3} + y^{2/3} = l^{2/3}.$$

Buckling leads to a decrease in the stress concentration in the neighborhood of the hole. For example, in the case of a slit of length $2l$, the stress concentration factor at the end of the slit decreases approximately to the value corresponding to a crack of length l emerging onto the free boundary of a half-plane perpendicular to the boundary.

Let now, at the points $z = \pm l$ of an infinite solid plate stretched at infinity by stresses $\sigma_y = \sigma_x = \sigma$, there act concentrated forces, equal in magnitude to X and directed toward one another. In this case the contour of the buckled zone is an ellipse (4)

$$x^2 + \lambda^{-2} y^2 = l^2 \qquad (\lambda = X/\pi\sigma l);$$

the stresses in the unbuckled region are determined by the Muskhilishvili potentials

$$\Phi(z) = \frac{\sigma\zeta}{2\sqrt{\zeta^2 - 1}}, \qquad \Psi(z) = l\Phi'(z)\left(\lambda\sqrt{\zeta^2 - 1} - \zeta\right),$$

$$z/l = \zeta + \lambda\sqrt{\zeta^2 - 1}, \tag{2}$$

and in the buckled region the stresses are equal to zero. It should be noted that for $\lambda \ll 1$ and $\lambda \gg 1$ solution (2) will not be suitable because of boundary-layer type effects.

§ 2. It is of interest to find the shapes of holes that ensure the maximum strength of a plate. The condition for maximum strength of a viscoelastic plate with a hole is the condition of equal strength of the hole, which requires a uniform distribution of the stress concentration along the contour of the hole $(3, 5)$. Suppose, for example, that the hole is in a homogeneous stress field

$$\sigma_x = \sigma_x^\infty, \qquad \sigma_y = \sigma_y^\infty, \qquad \tau_{xy} = 0,$$

and a constant external load is applied to the contour of the hole,

$$\sigma_n = p, \qquad \tau_{nt} = 0.$$

It then turns out $(5, 3)$ that the plate with a hole will possess maximum strength in the case when the contour of the hole belongs to the family of similar ellipses,

$$\frac{x^2}{(1+m)^2} + \frac{y^2}{(1-m)^2} = c^2, \qquad m = \frac{\sigma_y^\infty - \sigma_x^\infty}{2p - \sigma_y^\infty - \sigma_x^\infty}. \tag{3}$$

The stress σ_t on the contour of such holes, which determines the stress concentration, is equal to

$$\sigma_x^\infty + \sigma_y^\infty - p.$$

The indicated requirement of equal strength may sometimes prove structurally unacceptable. Therefore we shall solve the problem for a weaker condition, namely, assuming that on the sought contour of the hole L the boundary condition must be satisfied

$$\sigma_t = \sigma + \gamma \sin^2 \alpha \qquad (\sigma = \text{const}, \quad \gamma = \text{const}) \tag{4}$$

(nt is the direction of the normal and the tangent to the contour, forming a right-handed coordinate system; α is the angle between the axes x and n).

Thus, it is required to find the contour L and the Muskhelishvili potentials $\Phi(z)$ and $\Psi(z)$ from condition (4) and the conditions

$$\sigma_n = p, \qquad \tau_{nt} = 0 \quad \text{on } L;$$

$$\text{as } z \to \infty \qquad \Phi(z) = \frac{1}{4}(\sigma_y^\infty + \sigma_x^\infty) + O(z^{-2}), \tag{5}$$

$$\Psi(z) = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + O(z^{-2}).$$

Let us pass to the parametric plane ζ by means of the transformation $z = \omega(\zeta)$, $\omega(\zeta) = c_0\zeta + O(\zeta^{-1})$ as $\zeta \to \infty$. The analytic function $\omega(\zeta)$ defines a conformal mapping of the exterior of the unit circle of the ζ-plane onto the exterior of the unknown contour L of the physical z-plane; it is to be determined. The constant c_0 is, obviously, positive. Denote $\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$.

Conditions (4) and (5), using the known Kolosov–Muskhelishvili formulas [6], may be written in the ζ-plane in the form

$$4\,\mathrm{Re}\,\varphi(\zeta) = p + \sigma - \frac{\gamma\,[\zeta\omega'(\zeta) - \bar\zeta\,\overline{\omega'(\zeta)}]^2}{4|\omega'(\zeta)|^2} \qquad \text{for } |\zeta| = 1,$$

$$2\,\frac{\zeta^2\omega'(\zeta)}{\overline{\omega'(\zeta)}}\left[\frac{\varphi'(\zeta)}{\omega'(\zeta)}\,\overline{\omega(\zeta)} + \psi(\zeta)\right] = \sigma - p - \frac{\gamma\,[\zeta\omega'(\zeta) - \bar\zeta\,\overline{\omega'(\zeta)}]^2}{4|\omega'(\zeta)|^2} \qquad \text{for } |\zeta| = 1, \tag{6}$$

$$\text{as } \zeta \to \infty \qquad \omega(\zeta) = O(\zeta), \quad \varphi(\zeta) = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + O(\zeta^{-2}),$$

$$\psi(\zeta) = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + O(\zeta^{-2}).$$

To solve the nonlinear boundary-value problem (6), we apply the method of functional equations [7]. We write the boundary-value problem (6) in the form of a system of two functional equations valid in the entire ζ-plane:

$$2\varphi(\zeta)+2\bar{\varphi}\left(\frac{1}{\zeta}\right)=p+\sigma-\frac{\gamma}{4\omega'(\zeta)\bar{\omega}'(1/\zeta)}\left[\zeta\omega'(\zeta)-\frac{1}{\zeta}\,\bar{\omega}'\left(\frac{1}{\zeta}\right)\right]^2, \qquad (7)$$

$$2\zeta^2\omega'(\zeta)\left[\frac{\varphi'(\zeta)}{\omega'(\zeta)}\bar{\omega}\left(\frac{1}{\zeta}\right)+\psi(\zeta)\right]=$$

$$=\bar{\omega}'\left(\frac{1}{\zeta}\right)\left\{\sigma-p-\frac{1}{4\omega'(\zeta)\bar{\omega}'(1/\zeta)}\left[\zeta\omega'(\zeta)-\frac{1}{\zeta}\,\bar{\omega}'\left(\frac{1}{\zeta}\right)\right]\right\}.$$

We seek the function $\omega(\zeta)$ in the form of a polynomial

$$\omega(\zeta)=c_0\zeta+c_1\zeta^{-1}+c_3\zeta^{-3}+\ldots+c_{2n+1}\zeta^{-2n-1}. \qquad (8)$$

The constants $c_1, c_3, \ldots, c_{2n+1}$ are positive by the conditions of symmetry. We determine the number n by comparing the orders of the quantities in the right- and left-hand sides of the second functional equation (7) as $\zeta \to \infty$. We obtain $n=0$. Thus, $\omega(\zeta)$ is equal to

$$\omega(\zeta)=c_0\zeta+c_1/\zeta. \qquad (9)$$

From the first functional equation (7), using (9), one can find $\varphi(\xi)$

$$\varphi(\xi)=\frac{1}{4}(\sigma_x^\infty+\sigma_y^\infty)-\gamma(c_0^2-c_1^2)/8c_0(c_0\xi^2-c_1), \qquad (10)$$

where the condition must be satisfied

$$\frac{c_1}{c_0}=\frac{2}{\gamma}(\sigma_x^\infty+\sigma_y^\infty-p-\sigma)-1. \qquad (11)$$

The function $\psi(\xi)$ is determined by the second equation (7). From the condition at infinity it follows that

$$\gamma=\frac{(\sigma_x^\infty+\sigma_y^\infty-p-\sigma)(3p-\sigma-\sigma_x^\infty-\sigma_y^\infty)}{\sigma_y^\infty-\sigma_x^\infty+p-\sigma}.$$

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Received
17 XI 1965

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Note: Figure translations are in progress. See original paper for figures.