HAUSDORFF SPACES OF MINIMAL WEIGHT

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 519.50 + 519.54

MATHEMATICS

V. GOLDSHTEIN

HAUSDORFF SPACES OF MINIMAL WEIGHT

(Presented by Academician P. S. Aleksandrov on 18 IX 1965)

We shall consider only topological spaces satisfying Hausdorff' s separation axiom.

We shall say that a cover Γ **separates the points of the space** (X, T) if for any points $x', x'' \in X$ there exist sets $P', P'' \in \Gamma$ such that $x' \in P'$, $x'' \in P''$, $P' \cap P'' = \emptyset$.

Definition 1. We shall call a space (X, T) a **space of minimal weight** if the cardinality of any of its open covers that separates points is not less than the weight of this space.

It is easy to see that a space (X, T) is then and only then a space of minimal weight when:

1. The weight of any Hausdorff topology T_0 on the set X that is majorized by the topology T is not less than the weight of the topology T. All spaces with a countable base and all noncompactifiable spaces (or, in other words, minimal spaces), in particular all bicompacts, are spaces of minimal weight.

It is interesting that in condition 1 one cannot write "equal to" instead of "not less than,"since on a countable set there exists a topology T_0 without a countable base, whereas the countable discrete space is a space of minimal weight.

Let us note that

2. If a space contains a subspace of minimal weight equal to it in weight, then it itself is of minimal weight.

In general, if a space contains a subspace of minimal weight equal to τ, then the cardinality of any open cover of this space that separates points is not less than τ. Therefore

$2'$. If the weight of the space (X, T) is equal to $\sup\limits_{a \in A} \tau_a$, where τ_a is the weight of some subspace of minimal weight X_a of this space, $a \in A$, then the space (X, T) is a space of minimal weight.

From this we immediately obtain

3. If the space (X,T) is the Tychonoff product of spaces of minimal weight (X_a,T_a), $a\in A$, and the weight of one of them is not less than the cardinality of A, then (X,T) is a space of minimal weight.

Theorem 1. *The Tychonoff product of spaces of minimal weight is a space of minimal weight.*

Proof. Let a family (X_a,T_a), $a\in A$, of spaces of minimal weight be given; we may assume that each of them contains at least two distinct points. It remains for us to consider the case where the cardinality of the set A, say τ, is greater than the weight of any space (X_a,T_a), $a\in A$. But then the weight of the Tychonoff product of all these spaces is equal to τ, and hence is equal to the weight of the space D^τ (D is a two-point space), a homeomorphic image of which is contained in the product under consideration. Hence the latter is a space of minimal weight. The theorem is proved.

A. Arkhangel' skii introduced and studied the class of spaces whose weight is equal to the smallest cardinality of their networks (see ([1], [2])). We shall call them A-spaces. We shall say that a space is an A-space if its weight is equal to its c-weight, where the c-weight is defined as the least cardinality of its networks. In addition, define the n-weight of a space as the least cardinality of covers that separate points.

Lemma 1. *The n-weight of any space is not greater than its c-weight, and the latter, by A. Arkhangel' skii ([1]), is not greater than the weight of this space.*

Proof. Let a space (X,T) and its network C be given. The case when the set X is finite is trivial. Therefore suppose that the set X is infinite. Let Φ be the set of all pairs $B=\{B',B''\}\subseteq C$ for which there exists a pair $\varphi(B)=\{P',P''\}$ satisfying the condition

$$4.\quad P'\cap P''=\emptyset.$$

At the same time either the conditions $B'\subseteq P'$ and $B''\subseteq P''$, or the conditions $B'\subseteq P''$ and $B''\subseteq P'$, are satisfied.

Then the cardinality of the set Φ, as well as the cardinality of the family $\bigcup_{B\in\Phi}\varphi(B)$, is not greater than the cardinality of the network C.

If the family $\bigcup_{B\in\Phi}\varphi(B)$ separates points and is an open cover of the space (X,T), then the lemma is proved.

But for arbitrary distinct points $x',x''\in X$ there exist their disjoint neighborhoods H' and H'', which in turn contain sets $B',B''\in C$, for which $x'\in B'\subseteq H'$ and $x''\in B''\subseteq H''$. Hence the function φ is defined for the pair $\{B',B''\}$, i.e. condition 4 for the pair $\varphi(\{B',B''\})=\{P',P''\}$ is fulfilled.

From the fact that the n-weight of a space of minimal weight is equal to its weight, it follows that

Theorem 2. *Every space of minimal weight is an A-space.*

From condition 1 it is easy to derive theorem 3.

Theorem 3. *Every space can be compactified to some space of minimal weight.*

On the other hand, it can be proved that the space of rational numbers (with the usual topology) cannot be compactified to a minimal space.* This means that in theorem 3 spaces of minimal weight cannot be replaced by non-compactifiable spaces.

Institute of Mathematics
Polish Academy of Sciences
Warsaw, Poland

Received
14 IX 1965

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* A **non-compactifiable**, or **minimal**, space is a space whose topology T does not majorize any topology distinct from the topology T. By a **network** of a space (X, T) is understood a set C of subsets of the set X possessing the following property: if $x \in P \in T$, then there exists a set $B \in C$ for which $x \in B \subseteq P$.

Note: Figure translations are in progress. See original paper for figures.

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