# EQUIVALENCE OF TWO CLASSES OF MAPPINGS OF THE SPHERE

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## Abstract

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# EQUIVALENCE OF TWO CLASSES OF MAPPINGS OF THE SPHERE

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In this note we consider a class of mappings of the $n$-dimensional sphere onto itself, interest in which arose in connection with the hypothesis that mappings of this class coincide with mappings that can be approximated by homeomorphisms (see ([1]), p. 169). It is known that for $n = 1$ and 2 the class of mappings approximable by homeomorphisms coincides with the monotone mappings of the sphere onto the sphere ([2]). Already for $n = 3$ this is not so, as is shown by Bing' s example (([3]), p. 7) and by a more complicated construction of L. V. Keldysh (see ([1]), p. 169). On the other hand, in ([4]) it is shown that mappings that can be approximated by homeomorphisms have the following property:

$(K)$ The preimage of every cellular set is a cellular set.

A subset of $S^n$ is called cellular ([5]) if it is the intersection of a sequence of $n$-dimensional cells, in which each following one lies inside the preceding one. We shall call a mapping of $S^n$ into itself cellular if it has property $(K)$.

A mapping of $S^n$ was called pointlike by Bing if it has the following property:

$(T)$ The preimage of every point is a cellular set.

It is clear that property $(K)$ implies property $(T)$. We prove the converse:

**Theorem.** *For $n \neq 4$, a pointlike mapping of the sphere onto itself is cellular.*

For $n \leq 2$ every acyclic continuum in the sphere is cellular and, under monotone mappings of the sphere onto itself, the preimages of points are acyclic; therefore both classes of mappings coincide with the monotone mappings of the sphere onto itself and hence coincide with each other. For $n = 3$ and $n > 4$ we rely on the characterization of Euclidean space given by Edwards ([8]) and Stallings ([6]), respectively. In addition, in both cases we rely on Lemma 1 proved below, which is an application to the present situation of the technique used by Smale in the proof of the "homotopy theorem of Vietoris" ([7]).

**Lemma 0.** *Let $K$ be a continuum lying in the sphere $S^n$. The following properties of $K$ are equivalent:*

1. *$K$ is a cellular continuum.*
2. *$S^n \setminus K$ is homeomorphic to $R^n$.*
3. *The quotient space of the continuous decomposition of the sphere into $K$ and the points of the complement of $K$ is homeomorphic to $S^n$.*

The equivalence of 1 and 3 is established in ([5]), and the equivalence of 2 and 3 is obvious. We shall need the equivalence of 1 and 2.

**Proof of the theorem.** Let a pointlike mapping of the sphere onto itself be given, and let a cellular continuum $K$ in $S^n$ be given. We note that, by property 2 of Lemma 0 and Alexander duality, cellular continua are acyclic, and therefore, by the Vietoris theorem, the homology groups of $fS^n$ coincide with the homology groups of $S^n$. Hence, $fS^n = S^n$. We must prove that,

that $f^{-1}K$ is a cellular continuum. We shall again use the equivalence of properties 1 and 2 of Lemma 0 and show that $S^n \setminus K$ is homeomorphic to $R^n$. For this, by Stallings' criterion ([6]), it is enough to establish that all $\pi_i(S^n \setminus f^{-1}K) = 0$ and that $S^n \setminus f^{-1}K$ is unknotted at infinity, i.e., that for every compact set $C \subset S^n \setminus f^{-1}K$ there is a compact set $D \subset S^n \setminus f^{-1}K$, containing $C$, such that $S^n \setminus D \setminus f^{-1}K$ is connected and simply connected. The proofs of all these assertions are based on the application of the following lemma.

**Lemma 1.** Let $f : S^n \to S^n$ be a point mapping; $P$ a finite polyhedron; $Q$ a subpolyhedron, and let $\varphi : P \to S^n$, $\Psi : Q \to S^n$ be such that $\psi = f\Psi = \varphi|_Q$. Then for every $\varepsilon > 0$ there exists a mapping $\Phi : P \to S^n$ such that $\Phi|_Q = \Psi$ and $\rho(f\Phi, \varphi) < \varepsilon$, where $\rho$ is the metric in the space of mappings of $P$ into $S^n$.

We first show how the proof of the theorem is completed with the help of the lemma.

Let us show that $\pi_i(S^n \setminus f^{-1}K) = 0$. Let $\Psi : S^i \to S^n \setminus f^{-1}K$ be any continuous mapping of the boundary of an $(i+1)$-cell $B^{i+1}$, and let $\psi = f^\Psi$. Since $K$ is a cellular continuum and $S^n \setminus K$ is homeomorphic to $R^n$, there exists an extension $\varphi : B^{i+1} \to S^n \setminus K$ of the mapping $\psi$. By the lemma, for every $\varepsilon > 0$ there exists an extension $\Phi : B^{i+1} \to S^n$ of the mapping $\Psi$ such that $\rho(f\Phi, \varphi) < \varepsilon$. If $\varepsilon < d(\varphi(B^{i+1}), K)$, then $\Phi(B^{i+1})$ lies outside $f^{-1}K$, and since $\Psi : S^i \to S^n \setminus f^{-1}K$ was chosen arbitrarily, $\pi_i(S^n \setminus f^{-1}K) = 0$.

Let us show that $S^n \setminus f^{-1}K$ is unknotted at infinity. Let $C$ be an arbitrary compact set in $S^n \setminus f^{-1}K$, and let $V$ be such a cellular neighborhood of $K$ that $f^{-1}V \subset S^n \setminus C$. Put $D = f^{-1}(S^n \setminus V)$ and show that $\pi_1(S^n \setminus f^{-1}K \setminus D) = 1$. As above, let $\Psi : S^1 \to S^n \setminus f^{-1}K \setminus D$ be a mapping of the boundary $S^1$ of the disk $B^2$ into $S^n \setminus f^{-1}K \setminus D$, and let $\psi = f^\Psi$. It is clear that $\psi(S^1) \subset V \setminus K$. Since $n > 2$, $K$ is cellular and $V$ is a cellular neighborhood of $K$, there exists an extension $\varphi : B^2 \to V \setminus K$ of the mapping $\psi$. By the lemma, for every $\varepsilon = 0$ there exists such an extension $\Phi : B^2 \to S^n$ of the mapping $\Psi$ that $\rho(f\Phi, \varphi) < \varepsilon$. If $\varepsilon < d(\varphi(D^2), K \cup (S^n \setminus V))$, then $f\Phi(D^2) \subset S^n \setminus K$ and, consequently, $\Phi(D^2) \subset S^n \setminus f^{-1}K \setminus D$. Since $\Psi$ was arbitrary, $S^n \setminus f^{-1}K \setminus D$ is simply connected. In exactly the same way it is shown that $S^n \setminus f^{-1}K \setminus D$ is connected. Since $C$ was an arbitrary compact set, $S^n \setminus f^{-1}K$ is unknotted at infinity. The theorem is proved.

**Proof of Lemma 1.** We shall use induction on the dimension of the polyhedron. As usual in such cases, it is enough to consider the situation when $P$ is a $k$-dimensional cell and $Q$ is its boundary. Put $P = B^k,\ Q = S^{k-1}$. If $k = 0$, then $B^k$ is a point, and for $\Phi(B^k)$ one should take an arbitrary point in $f^{-1}(\varphi B^k)$. Then $f\Phi = \varphi$. Suppose that in dimensions less than $k$ the lemma is true. Let $\varepsilon > 0$ be given. For each point $x \in S^n$ there is a neighborhood $U(x)$ such that $f^{-1}(\overline{Ux})$ lies in such a cellular neighborhood $V(x)$ that the diameter of $f(V(x))$ is less than $\varepsilon/3$. Choose from the sets $U(x)$ a finite covering $\{U_i\}$ and denote by $\delta$ the minimum of $\varepsilon/3$ and the Lebesgue number of this covering. Using the inductive assumption, we can construct an extension $\Psi^{k-1}$ of the mapping $\Psi$ to the $(k-1)$-dimensional skeleton so fine a triangulation $T$ of the cell $B^k$ that the image of each simplex of $T$ has diameter less than $\delta/3$. In doing so the extension $\Psi^{k-1}$ may be chosen so that the requirement

$$\rho(f\Psi^{k-1}, \varphi|_{|T|^{k-1}}) < \delta/3,$$

is satisfied, where $|T|^{k-1}$ is the $(k-1)$-skeleton of $T$. The mapping $\Psi^{k-1}$ has the property that the diameter of the image of the boundary of each $k$-simplex of the triangulation $T$ is less than $\delta$, and therefore each such image of the boundary of a simplex lies in some neighborhood $U_i$. By assumption, $f^{-1}(U_i)$ lies in the cell $V_i$, the diameter of the image of which is less than $\varepsilon/3$. Let $\Delta^k$ be an arbitrary $k$-simplex of $T$. $\Psi^{k-1}(\dot{\Delta}^k)$ lies in some $f^{-1}(U_i)$ and, consequently, in some cell $V_i$. Hence one can construct an extension $\Phi|_{\Delta^k}$ of the mapping $\Psi^{k-1}$, for which the image $\Phi(\Delta^k)$ lies in $V_i$. Thus $\Phi$ can be constructed on the whole cell $B^k$. We shall show that $\rho(f\Phi, \varphi) < \varepsilon$.

Indeed, if the point $x \in \Delta^k$, where $\Delta^k$ is some closed simplex of the triangulation, then the diameters of $\varphi(\Delta^k)$ and $f\Phi(\Delta^k)$ are less than $\varepsilon/3$, and the distance between the images of the boundary of $\Delta^k$ under the mappings $\varphi$ and $f\Phi|_{\Delta^k} = f\Psi^{n-1}$ is also less than $\varepsilon/3$. Therefore the distance between $\varphi(x)$ and $f\Phi(x)$ is less than $\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. The lemma is proved.

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*Note: Figure translations are in progress. See original paper for figures.*