EMBEDDING THEOREMS FOR THE SPACES $W_{p_0\bar p}^{(\bar l)}$, $B_{p_0\bar\theta}^{(\bar l)}$

A. D. BERIEV

1966

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

A. D. BERIEV

EMBEDDING THEOREMS FOR THE SPACES $W^{(\bar{l})}_{p_0\bar{p}}$, $B^{(\bar{l})}_{p_0\bar{\theta}}$

(Presented by Academician S. L. Sobolev on 18 V 1965)

In the present note embedding theorems are proved for the spaces

$$W^{(\bar{l})}_{p_0\bar{p}}(E^n), \quad B^{(\bar{l})}_{p_0\bar{\theta}}(E^n),$$

generalizing the results of V. P. Il' in ([4]) and A. Kh. Gudiyev ([5]); l_i $(i = 1, \dots, s)$ are arbitrary positive numbers.

Let $E^\mu, E^{n_j}, E^{n(i)}$ be, respectively, μ-, n_j-, and $n^{(i)}$-dimensional Euclidean spaces of points $\bar{t}(t_1, t_2, \dots, t_\mu)$, $\bar{x}^j(x^j_1, \dots, x^j_{n_j})$, $\bar{y}^{(i)}(y^{(i)}_1, \dots, y^{(i)}_{n(i)})$;

$$n = \sum_{i=1}^{s} n^{(i)} = \sum_{j=1}^{\tau} n_j; \quad E^{n(i)} \cap E^{n_j} = E^{m^{(i)}_j}; \quad r = r(\bar{x}, \bar{y}, \bar{t}) = \left[\sum_{i=1}^{s} r_i^{2/\chi_i} + |\bar{t}|^{2/\chi_0}\right]^{1/2},$$

where

$$r_i = \left[|\bar{y}^{(i)} - \bar{x}^{(i)}|\right]^{1/2} = \left[\sum_{k=1}^{n(i)} (y^{(i)}_k - x^{(i)}_k)^2\right]^{1/2}; \quad \chi_i > 0 \ (i = 0, 1, \dots, s); \quad D_{0;s}$$

$$= {}^{\mu}_{h^{\chi_0}}(\bar{0}) \times \prod_{i=1}^{s} {}^{n(i)}_{h^{\chi_i}}(\bar{x}^{(i)}); \quad {}^{\mu}_{h^{\chi_0}}(\bar{0}) \subset E^\mu$$

is the μ-dimensional ball of radius h^{χ_0} with center at $(\bar{0})$; ${}^{n(i)}_{h^{\chi_i}}(\bar{x}^{(i)}) \subset E^{n(i)}$ is the $n^{(i)}$-dimensional ball of radius h^{χ_i} with center at $(\bar{x}^{(i)})$; $(\bar{l}) = (l_1, \dots, l_s)$; $\bar{p} = (p_1, \dots, p_s)$; $\theta = (\theta_1, \dots, \theta_n)$; $s \le n$; $(\rho_{0;s}) = (\rho_0, \rho_1, \dots, \rho_s)$; $(\bar{q}) = (q_1, \dots, q_\tau)$,

$$A_{(D_{0;s})}^{(\rho_{0;s})}[|f|^{\rho_s}] \equiv \|f\|_{L_{(\rho_{0;s})}(D_{0;s})}.$$

Theorem 1. If

$$F(\bar{y},\bar{t}) \in L_{(\rho_{0;s})}(E^{n+\mu}),$$

$$1 \le \rho_i \le q_j < \infty \quad (i = 0, 1, \dots, s;\ j = 1, \dots, \tau); \quad \beta > 0,$$

$$\gamma > -\mu\rho_s/\rho_s'\rho_0,$$

$$\lambda = \frac{\rho_s}{\rho_s'}\left(\sum_{i=1}^{s}\frac{n^{(i)}\chi_i}{\rho_i} + \frac{\mu\chi_0}{\rho_0}\right) + \sum_{j=1}^{\tau}\frac{1}{q_j}\sum_{i=1}^{s} m_j^{(i)}\chi_i,$$

then

$$\left\|\left(A_{(D_{0;s})}^{(\rho_{0;s})}\left[\frac{|\bar{t}|^{\gamma}F(\bar{y},\bar{t})}{\left(\sqrt{r^2+H^2}\right)^{\lambda+\gamma\chi_0+\alpha}}\right]\right)^{\rho_s}\right\|_{L_{(\bar{q})}(E^n)} \le$$

$$\le \begin{cases} c_1 h^{\beta}\|F\|_{L(\bar{\rho}_0;s)(E^{n+\mu})}, & \text{if } \alpha = -\beta,\ H = 0, \\ c_2 H^{-\beta}\|F\|_{L(\bar{\rho}_0;s)(E^{n+\mu})}, & \text{if } \alpha = \beta,\ 0 < H \le h, \\ c_3\left\|\dfrac{|\bar{t}|^{\gamma}F(\bar{y},\bar{t})}{\left(\sqrt{|\bar{t}|^{2/\chi_0}+H^2}\right)^{\gamma\chi_0}}\right\|_{L(\bar{\rho}_0;s)(E^n\times\Pi^{\mu}_{h\chi_0}(\bar{0}))}, & \text{if } \alpha = 0,\ \gamma \ge 0,\ 0 < H \le h, \end{cases}$$

where c_1, c_2, c_3 are constants independent of H, h, F.

Theorem 2. If

$$F(\bar{y},\bar{t}) \in L_p(E^{n+\mu}), \qquad 1 < p < q_j < \infty \quad (j = 1, \dots, \tau),$$

$$\lambda = \frac{\mu\chi_0}{p'} + \sum_{j=1}^{\tau}\left(\frac{1}{p'} + \frac{1}{q_j}\right)\sum_{i=1}^{s} m_j^{(i)}\chi_i,$$

then

$$\left\|\int_{E^{\mu}} d\bar{t}\int_{E^n}\frac{F(\bar{y},\bar{t})}{r^{\lambda}}\,d\bar{y}\right\|_{L(\bar{q})(E^n)} \le c\,\|F\|_{L_p(E^{n+\mu})}, \tag{1}$$

where c does not depend on F.

Proof. On the basis of the theorem of Benedek and Panzone ([3]), the proof of the inequality

$$\int_{E_x^n}\int_{E_y^n}\int_{E_t^\mu}\frac{\varphi(\bar{x})F(\bar{y},\bar{t})}{r^\lambda}\,d\bar{t}\,d\bar{y}\,d\bar{x}\le c_1\|F\|_{L_p(E^{n+\mu})}\|\varphi\|_{L(\bar{q}')(E^n)} \tag{2}$$

is equivalent to the proof of inequality (1).

The following estimate holds:

$$\int_{E_t^\mu}\frac{|F(\bar{y},\bar{t})|}{r^\lambda}\,d\bar{t}\le c_2\,\bar{r}^{-\sum_{j=1}^{\tau}(1/p'+1/q_j)\sum_{i=1}^{s}m_j^{(i)}\chi_i}\,\|F\|_{L_p(E^\mu)}, \tag{3}$$

where

$$\bar{r}^2=\sum_{i=1}^{s}r_i^{2/\chi_i}>c_3\sum_{j=1}^{\tau}\sum_{i=1}^{s}r_{j,i}^{2/\chi_i},\qquad r_{j,i}=\sum_{k=\sum_{\eta=1}^{j-1}m_\eta^{(i)}+1}^{\sum_{\eta=1}^{j}m_\eta^{(i)}}\big(x_k^{(i)}-y_k^{(i)}\big)^2.$$

If inequality (3) is taken into account, then

$$\int_{E_x^n}\int_{E_y^n}\int_{E_t^\mu}\frac{\varphi(\bar{x})F(\bar{y},\bar{t})}{r^\lambda}\,d\bar{t}\,d\bar{y}\,d\bar{x}\le$$

$$\le c_4\int_{E_x^{m_\tau^{(1)}}}\int_{E_y^{m_\tau^{(1)}}}\cdots\int_{E_x^{m_\tau^{(s)}}}\int_{E_y^{m_\tau^{(s)}}}\int_{E_x^{m_1^{(1)}}}\int_{E_y^{m_1^{(1)}}}\cdots$$

$$\cdots\int_{E_x^{m_1^{(s)}}}\int_{E_y^{m_1^{(s)}}}\frac{\varphi(\bar{x})\|F\|_{L_p(E^\mu)}}{\left(\sum_{j=1}^{\tau}\sum_{i=1}^{s}r_{j,i}^{2/\chi_i}\right)^{\frac{1}{2}\left[\sum_{j=1}^{\tau}(1/p'+1/q_j)\sum_{i=1}^{s}m_j^{(i)}\chi_i\right]}}\,d\bar{y}_1^{(s)}\,d\bar{x}_1^{(s)}\cdots$$

$$\cdots d\bar{y}_1^{(1)}\,d\bar{x}_1^{(1)}\cdots d\bar{y}_\tau^{(s)}\,d\bar{x}_\tau^{(s)}\cdots d\bar{y}_\tau^{(1)}\,d\bar{x}_\tau^{(1)}. \tag{4}$$

If Sobolev's theorem ([1]) is applied $s-\tau$ times to the right-hand side of inequality (4), we obtain the required result.

On the basis of Theorems 1 and 2, with the aid of the integral inequalities of V. P. Il' in ([4]), the following embedding theorems are proved for E^n.

Theorem 3. *If $f\in W_{p_0\bar{p}}^{(\bar{l})}(E^n)$, $\nu_j^{(i)}$ $(i=1,\dots,s;\ j=1,\dots,n(i))$ are nonnegative integers satisfying the conditions*

$$\nu\sum_{i=1}^{s}\nu^{(i)},\qquad \nu^{(i)}=\sum_{j=1}^{n(i)}\nu_j^{(i)},\qquad 1\le p_i\le q_j<\infty\quad(i=0,1,\dots,s;\ j=1,\dots,\tau),$$

$$\varepsilon = 1 - \sum_{i=1}^{s} \frac{n^{(i)}}{l_i p_i} - \sum_{i=1}^{s} \chi_i \nu^{(i)} + \sum_{j=1}^{\tau} \frac{1}{q_j} \sum_{i=1}^{s} m_j^{(i)} \chi_i \geq 0, \qquad h > 0,$$

then:

$$1. \quad \|D_x^\nu f(\bar{x})\|_{L_{\bar{q}}(E^n)} \leq c_1 \|f\|_{W^{(\bar{l})}_{p_0\bar{p}}(E^n)}, \qquad \text{if } \varepsilon > 0.$$

$$2. \quad \|D_x^\nu f(\bar{x})\|_{L_{\bar{q}}(E^n)} \leq c_2 \|f\|_{L^{(\bar{l})}_{\bar{p}}(E^n)}, \qquad \text{if } \varepsilon = 0,\ 1 < p_i < q_j < \infty$$

$$(i = 1, \dots, s;\ j = 1, \dots, \tau).$$

Theorem 4. *If*

$$f \in B^{(\bar{l})}_{p_0\bar{p},\bar{\theta}}(E^n), \qquad 1 \leq p_i \leq \theta_i < \infty\ (i = 1, \dots, n), \qquad 1 \leq p_i \leq q_j < \infty$$

$$(i = 0, 1, \dots, n;\ j = 1, \dots, \tau),$$

$$\varepsilon = 1 - \sum_{i=1}^{n} \frac{1}{l_i p_i} - \sum_{i=1}^{n} \nu^{(i)} \chi_i + \sum_{j=1}^{\tau} \frac{1}{q_j} \sum_{i=1}^{n} \chi_i \geq 0,$$

then:

$$1. \quad \|D_x^\nu f(\bar{x})\|_{L_{\bar{q}}(E^n)} \leq c_1 \|f\|_{B^{(\bar{l})}_{p_0\bar{p},\bar{\theta}}(E^n)}, \qquad \text{if } \varepsilon > 0.$$

$$2. \quad \|D_x^\nu f(\bar{x})\|_{L_{\bar{q}}(E^n)} \leq c_2 \|f\|_{\mathcal{L}^{(\bar{l})}_{\bar{p}}(E^n)}, \qquad \text{if } \varepsilon = 0,\ p_i = \theta_i,\ 1 < p_i < q_j < \infty$$

$$(i = 1, \dots, n;\ j = 1, \dots, \tau).$$

We indicate the path of the proof of Theorem 4. For the proof we use inequality (29.2) of V. P. Il’ in ([4]) for the domain

$$D_{1;s} = \prod_{i=1}^{s} \Pi^{n(i)}_{h\chi_i}(\bar{x}^{(i)}).$$

Applying Minkowski’ s inequality to (29.2), we shall have

$$
\begin{aligned}
\|D_x^\nu f(\bar{x})\|_{L_{\bar{q}}(E^n)} \le C_3 h^{-\sum_{i=1}^n \chi_i(1+\nu^{(i)})} \left\| \int_{D_{1;s}} |f(\bar{y})|\, d\bar{y} \right\|_{L_{\bar{q}}(E^n)} \\
+ \sum_{i=1}^n \left\| \int_{I_{h\chi_i}(0)} dt \int_{D_{1;s}} \frac{\left|\Delta_i^2(t/2) D_{y^{(i)}}^{l_i} f(\bar{y})\right|}{r^{\lambda_i}}\, d\bar{y} \right\|_{L_{\bar{q}}(E^n)} .
\end{aligned}
\tag{5}
$$

The estimate holds

$$
h^{-\sum_{i=1}^n \chi_i(1+\nu^{(i)})} \left\| \int_{D_{1;s}} |f|\, dy \right\|_{L_{\bar{q}}(E^n)} \le c_4 h^{-\delta} \|f\|_{L_{p_0}(E^n)}, \tag{6}
$$

where

$$
\delta = 1 - \varepsilon + \sum_{i=1}^n \frac{1}{l_i} \left(\frac{1}{p_0} - \frac{1}{p_i} \right).
$$

If $\varepsilon > 0$, then, on the basis of Theorem 1, for $\chi_0 = \chi_i$, $\mu = 1$, $\beta = \varepsilon$, $\rho_j = p_i$ $(j = 0, 1, \dots, s)$,

$$
\bar{\lambda} = \lambda = \frac{1}{p_i} \left(\sum_{i=1}^n \chi_i + \chi_i \right) + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{i=1}^n \chi_i
$$

after simple transformations we obtain the estimate

$$
\left\| \int_h^{\chi_i(\bar{0})} dt \int_{D_{1;s}} \frac{F(\bar{y}, \bar{t})\, |\bar{t}|^\gamma}{r^{\bar{\lambda} + \gamma\chi_i - \varepsilon}}\, d\bar{y} \right\|_{L_{(\bar{q})}(E^n)} \le c_5 h^\varepsilon \, \|f\|_{\mathcal{L}^{l_i}_{p_i, y^{(i)}}(E^n)}. \tag{7}
$$

If $\varepsilon = 0$, then, on the basis of Theorem 2, we obtain the estimate

$$
\left\| \int_h^{\chi_i(\bar{0})} dt \int_{D_{1;s}} \frac{F(\bar{y}, \bar{t})}{r^{\bar{\lambda}}}\, d\bar{y} \right\|_{L_{(\bar{q})}(E^n)} \le c_6 \, \|f\|_{\mathcal{L}^{l_i}_{p_i, y^{(i)}}(E^n)}. \tag{8}
$$

From inequalities (5), (6), (7), and (8) we obtain

$$\left\| D_{\bar{x}}^{\nu} f(\bar{x}) \right\|_{L_{(\bar{q})}(E^n)} \leq \begin{cases} c_7 \left(h^{-\delta} \|f\|_{L_{p_0}(E^n)} + h^{\varepsilon} \|f\|_{\mathcal{L}^{\bar{l}}_{\frac{\bar{p}}{\bar{\theta}}}(E^n)} \right), & \text{if } \varepsilon > 0, \\ c_8 \left(h^{-\delta} \|f\|_{L_{p_0}(E^n)} + \|f\|_{\mathcal{L}^{\bar{l}}_{\bar{p}}(E^n)} \right), & \text{if } \varepsilon = 0,\ \theta_i = p_i, \\ 1 < p_i < q_j < \infty \quad (i = 1, \dots, n;\ j = 1, \dots, \tau). \end{cases} \tag{9}$$

(10)

If for the right-hand side of (9) one finds the minimum of the function for the corresponding h, and in (10) lets $h \to \infty$, then we obtain what was required.

North Ossetian State
Medical Institute

Received
13 V 1965

REFERENCES

1. S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, L., 1950.
2. S. L. Sobolev, *Matem. sborn.*, 4 (46), No. 3, 471 (1938).
3. A. Benedek, R. Panzone, *Duke Math. J.*, 28, No. 3 (1961).
4. V. P. Il' in, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, 66 (1962).
5. A. Kh. Gudiev, *DAN*, 160, No. 2 (1965).
6. A. Kh. Gudiev, *DAN*, 149, No. 2 (1963).