INVESTIGATION OF GENERAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC EQUATIONS BY THE METHOD OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract

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INVESTIGATION OF GENERAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC EQUATIONS BY THE METHOD OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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1. Boundary-value problems for quasilinear parabolic equations of order $2m$ in cylindrical domains with homogeneous* boundary conditions can be reduced to the Cauchy problem

$$v'(t) + A(t, v)v = f(t, v) \quad (0 < t \leq T), \qquad v(0) = v_0, \tag{1}$$

in some Banach space E with a strongly positive operator A, generated by an elliptic differential expression and a system of boundary conditions (see, for example, the survey ([1])).

An operator A, acting in a Banach space E, having everywhere dense domain of definition $D[A]$, is called strongly positive (s.p. E-operator) if for any λ with $\operatorname{Re}\lambda \geq 0$ the operator $A + \lambda I$ has a bounded inverse and $\|(A + \lambda I)^{-1}\|_E \leq C(|\lambda| + 1)^{-1}$. As is known (see ([2,3])), elliptic operators are s.p. L_p-operators for any $p \in (1, \infty)$.

If the coefficients of the boundary conditions depend on t, then the s.p. L_p-operator A generated by them will have a domain of definition depending on t. The existence theorems known up to now for the abstract problem (1) have made it possible to obtain existence theorems for solutions of boundary-value problems with variable boundary conditions only for equations of second order.**

In the present article a new method is presented for investigating problem (1), which has made it possible to obtain existence theorems for classical solutions of general normal (see ([5])) boundary-value problems for parabolic equations of arbitrary order. The homogeneous linear problem studied is

$$v'(t) + A(t)v = 0 \quad (0 \leq \tau < t \leq T), \qquad v(\tau) = v_0, \tag{2}$$

for whose solutions a number of estimates of the same kind are established as in the case of equations with operators having a constant domain of definition. These estimates make it possible, by a known method (see ([1])), to obtain existence theorems also for the general problem (1). In another paper by the author it will be shown how the abstract technique developed here makes it possible to prove an existence theorem for a general quasilinear parabolic equation with general quasilinear normal boundary conditions.

2. Introduce the following notation

$$M_i(t,\tau;v,w) = |t-\tau|^{\alpha_i} \|A(t)v\|_E^{\beta_i} \|v\|_E^{1-\beta_i} \|A^*(\tau)w\|_{E^*}^{\gamma_i} \|w\|_{E^*}^{1-\gamma_i},$$

$$N_i(t,\tau;v,w) = |t-\tau|^{\xi_i} \|A(t)v\|_E \|w\|_{E^*},$$

$$R_i(t,\tau;v,w) = |t-\tau|^{\eta_i} \|v\|_E \|A^*(\tau)w\|_{E^*}.$$

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* A problem with nonhomogeneous boundary conditions can, as is known, be reduced by a change of functions to a problem with homogeneous conditions.

** The method applied in ([4]) made it possible to consider only boundary-value problems reducible to the first boundary-value problem.

Here A^* is the operator acting in the adjoint space E^* and adjoint to the operator A. Let $0 < \alpha_i \leqslant 1$, $0 < \gamma_i < \alpha_i$, $0 < \beta_i \leqslant 1-\gamma_i$, $0 < \varepsilon_i \leqslant 1$, $0 < \eta_i \leqslant 1$. Finally, put

$$\varepsilon = \min_i \min(\alpha_i - \gamma_i, \varepsilon_i), \quad \varepsilon' = \min_i \min(\alpha_i - \beta_i, \eta_i), \quad \rho = \min_i (1-\gamma_i), \quad \rho' = \min_i (1-\beta_i), \quad h = \min_i \min(\alpha_i, \varepsilon_i),$$

Theorem 1. Let $A(t)$ for each $t \in [0,T]$ be an s.p. E-operator. Suppose that for any $t, \tau \in [0,T]$, $v \in D[A(t)]$, $w \in D[A^*(\tau)]$, the inequality

$$\big|[A(t)v, w] - [v, A^*(\tau)w]\big| \leqslant \sum_{i=1}^{r} B_i(t,\tau;v,w), \tag{3}$$

holds, where $[\varphi,\psi]$ is the value of the functional $\psi \in E^*$ on the element $\varphi \in E$; r is a positive integer; B_i are seminorms defined on $D[A(t)] \times [A^*(\tau)]$ and satisfying the inequalities

$$B_i \leqslant c_i \min(M_i + N_i, M_i + R_i), \tag{4}$$

where c_i are constants. Then there exists an operator-function $U(t,\tau)$, defined for $0 \leqslant \tau \leqslant t \leqslant T$ with values in the space of bounded linear operators over E, possessing the following properties:

1) $U(t,\tau)$ is uniformly continuous jointly in t and τ for $t > \tau$, and strongly continuous for $t \geqslant \tau$.

2) $U(t,t) = I$, and for any $0 \leqslant \tau \leqslant s \leqslant t \leqslant T$ the identity

$$U(t,\tau) = U(t,s)U(s,\tau). \tag{5}$$

3) $U(t,\tau)$ for $t > \tau$ is uniformly and continuously differentiable with respect to t, and

$$U'_t(t,\tau) + A(t)U(t,\tau) = 0. \tag{6}$$

4) Problem (2), for any $v_0 \in E$, has a unique solution continuous for all $t \geqslant \tau$ and continuously differentiable for $t > \tau$:

$$v_\tau(t) = U(t,\tau)v_0. \tag{7}$$

If $v_0 \in D[A(\tau)]$, then the function $v_\tau(t)$ is continuously differentiable also for $t \geqslant \tau$ and satisfies equation (2). Here the derivative at the point $t = \tau$ is understood as the right derivative.

5) For any $0 \leqslant \tau \leqslant t \leqslant T$, $0 \leqslant \mu \leqslant \nu \leqslant \nu_0 < 1 + \varepsilon$, the inequality

$$\|A^\nu(t)U(t,\tau)A^{-\mu}(\tau)\|_E \leqslant c(\nu_0)|t-\tau|^{\mu-\nu} \tag{8}$$

holds.

6) If $\varepsilon' > 0$, then $U(t,\tau)$ is uniformly and continuously differentiable with respect to τ for $\tau < t$, and

$$U'_\tau(t,\tau) - \overline{U(t,\tau)A(\tau)} = 0, \tag{9}$$

where the bar denotes the closure of the operator in E.

7) For any $0 \leqslant \tau \leqslant t \leqslant T$, $0 \leqslant \mu \leqslant \nu \leqslant \nu_0 < 1 + \varepsilon'$, the inequality

$$\|\overline{A^{-\mu}(t)U(t,\tau)A^\nu(\tau)}\|_E \leqslant c(\nu_0)|t-\tau|^{\mu-\nu} \tag{10}$$

holds.

8) The operator $U(t,\tau)$ can be represented by a multiplicative integral.

This theorem generalizes Theorem 1 of ([6]). As in the proof of Theorem 1 of ([7]), the operator $U(t,\tau)$ is sought here in the form of a special multiplicative integral. In proving the convergence of the indicated process and in deriving estimates for the operator $U(t,\tau)$, a number of auxiliary assertions are used concerning semigroups generated by s.p. E-operators and concerning fractional powers of such operators. As is known ([8, 9]), s.p. E-operators generate analytic semigroups whose norms decrease exponentially. Fractional powers of s.p. E-operators have been studied in ([10, 11]).

Lemma 1. Let $t, s, \xi, \eta \in [0,T]$ and $\tau > 0$. Then for any $\mu, \nu \in [0,\rho)$ the inequality

$$\|A^\mu(\xi)\left[\exp\{-\tau A(t)\} - \exp\{-\tau A(s)\}\right]A^{-\nu}(\eta)\|_E \leq c\exp\{-\delta\tau\}\tau^{\nu-\mu}|t-s|^h, \tag{11}$$

holds; for any $\mu, \nu \in [0, \rho')$ the inequality

$$\|A^{-\nu}(\xi)\,[\exp\{-\tau A(t)\} - \exp\{-\tau A(s)\}]\,A^{\mu}(\eta)\|_E \le c\exp\{-\delta\tau\}\tau^{\nu-\mu}|t-s|^h \quad (12)$$

holds. Here $\delta > 0$.

From this the following follow successively:

Lemma 2. The inequalities

$$\|A^{\mu}(\xi)\,[A^{-\nu}(t) - A^{-\nu}(\tau)]\|_E \le c(\nu-\mu)|t-\tau|^h \quad (\mu \in [0,\rho'),\ \nu > \mu), \quad (13)$$

$$\|[A^{-\nu}(t) - A^{-\nu}(\tau)]\,A^{\mu}(\xi)\|_E \le c(\nu-\mu)|t-\tau|^{h'} \quad (\mu \in [0,\rho),\ \nu > \mu), \quad (14)$$

hold.

From Lemma 2, in particular, it follows that $D[A^{\mu}(\xi)]$ contains $D[A^{\nu}(\eta)]$ for any $\mu \in [0,\rho)$, $\nu > \mu$, and $\xi, \eta \in [0,T]$. This circumstance is decisive in the proof of Theorem 1 (cf. (12)).

Lemma 3. The inequalities $(0 \le \tau \le t \le T)$

$$\|A^{\mu}(t)\exp\{-(t-\tau)A(t)\}A^{-\nu}(\tau)\|_E \le c\exp\{-\delta(t-\tau)\}|t-\tau|^{\nu-\mu} \quad (0 \le \nu \le \mu \le 1+\varepsilon), \quad (15)$$

$$\|A^{-\nu}(t)\exp\{-(t-\tau)A(\tau)\}A^{\mu}(\tau)\|_E \le c\exp\{-\delta(t-\tau)\}|t-\tau|^{\nu-\mu} \quad (0 \le \nu \le \mu \le 1+\varepsilon') \quad (16)$$

hold.

In the same way as Theorem 2 follows from (13), one proves

Lemma 4. Let A be an s.p. E-operator and let $B(v,w)$ be a semilinear form, defined on $D[A] \times D[A^*]$ and satisfying, for some $0 < \alpha < 1$, $0 < \beta < 1$, the inequality

$$B(v,w) \le c\|Av\|_E^{\alpha}\|v\|_E^{1-\alpha}\|A^*w\|_{E^*}^{\beta}\|w\|_{E^*}^{1-\beta}. \quad (17)$$

Then for any

$$0 < \lambda_2 < \alpha < 1-\lambda_1 < 1, \quad 0 < \delta_2 < \beta < 1-\delta_1 < 1$$

the inequality

$$B(v,w) \le c\cdot c_1\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\frac{\delta_1+\delta_2}{\delta_1\delta_2}\times$$
$$\times \left\|A^{\alpha+\lambda_1}v\right\|_E^{\lambda_2/(\lambda_1+\lambda_2)}\left\|A^{\alpha-\lambda_2}v\right\|_E^{\lambda_1/(\lambda_2+\lambda_1)}\left\|(A^*)^{\beta+\delta_1}w\right\|_{E^*}^{\delta_2/(\delta_1+\delta_2)}\left\|(A^*)^{\beta-\delta_2}w\right\|_{E^*}^{\delta_1/(\delta_1+\delta_2)}. \quad (18)$$

holds.

Let us note that although A^* is not an s.p. E^*-operator ($D[A^*]$ may be non-dense in E^*), its fractional powers can be defined by the same formulas, and they will possess all the necessary properties of s.p. E^*-operators.

Theorem 1 makes it possible to investigate the linear nonhomogeneous equation and the quasilinear equation (1) and to obtain the same results as in Theorems 2 and 3 of (6).

3. We shall now show how, for elliptic operators in the spaces L_p, the conditions of Theorem 1 are verified.

Let Ω be a domain of n-dimensional Euclidean space with sufficiently smooth boundary S. Let a be an elliptic differential operator of order $2m$, and let a_{k_j}, $j = 1, \dots, m$, be a system of normal boundary operators of orders k_j, covering a. If the coefficients of a and a_{k_j} are sufficiently smooth, then the formal adjoints a^+ and $a^+_{k_j}$ are defined. The $a^+_{k_j}$ are normal and cover a^+ (see (5)). We supplement the system a_{k_j} to a Dirichlet system a_j, $j = 0, 1, \dots, 2m-1$. Then the system $a^+_{k_j}$ is supplemented to the corresponding Dirichlet system a^+_j, $j = 0, 1, \dots, 2m-1$.

Integration by parts proves

Theorem 2. For any $l = 1, \dots, 2m-1$ the identity holds

$$\int_\Omega av\cdot u\,dx = \sum_{i,j}\int_\Omega a(i)u\cdot a^+(j)v\,dx + \sum_{k=0}^{l-1}\int_S a_k u\cdot a^+_k(l)v\,dx + \sum_{k=l}^{2m-1}\int_S a_k u\cdot a^+_{2m-1-k}v\,dx, \tag{19}$$

where $a(i), a^+(j), a^+_k(l)$ are differential operators of orders $i \le 2m-l$, $j \le l$ and $2m-l-1$, respectively.

Let A and B be elliptic operators in $L_p(\Omega)$ $(1 < p < \infty)$, defined by elliptic expressions a and b of order $2m$ and by systems of homogeneous normal boundary conditions of the same order. Identity (19) makes it possible to estimate

$$\Delta = \left|\int_\Omega Au\cdot v\,dx - \int_\Omega u\cdot B^* v\,dx\right| \quad (u \in D[A],\ v \in D[B^*])\,. \tag{20}$$

It must be taken into account that either a_{2m-1} and b_{2m-1} enter into the system of boundary conditions for A and B, or a^+_0 and b^+_0 enter into the system of boundary conditions for A^* and B^*. Therefore, in estimating Δ one has to deal with integrals over S containing normal derivatives to S only up to order $2m-2$, which makes it possible, for operators $A(t)$ depending on the parameter t, to obtain estimates (3), (4). In doing so, known coercivity inequalities for elliptic operators and multiplicative inequalities for norms in the spaces W^l_p are used (see, for example, (4)).

Because of lack of space we do not give exact restrictions on smoothness with respect to t and x. We note only that the minimal restrictions on smoothness with respect to x are obtained when identity (19) is used with $l = 1$.

Finally, suppose that for any $t \in [0, T]$, $u \in D[A(t)]$, and λ with $\operatorname{Re}\lambda \ge 0$, the inequality

$$\|A(t)u + \lambda u\|_{L_p} \ge c_p(t)\|u\|_{W^{2m}_p} \tag{21}$$

holds. For conditions under which (21) holds, see (2, 3). Then $A(t)$ will be an s.p. L_p-operator.

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Note: Figure translations are in progress. See original paper for figures.