# FORMULATION OF THE PROBLEM OF FLOW PAST BODIES WITH JETS AND AN EXACT SOLUTION OF TWO CLASSES OF PROBLEMS IN AN IDEAL FLUID

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## Abstract

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V. M. SHURYGIN

# FORMULATION OF THE PROBLEM OF FLOW PAST BODIES WITH JETS AND AN EXACT SOLUTION OF TWO CLASSES OF PROBLEMS IN AN IDEAL FLUID

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## I. Formulation of the problem.

If, in the case of flow past bodies with jets, at subsonic flow velocities one specifies, in addition to the contour of the body and the flow at infinity, also the velocities at the nozzle sections, then such a problem turns out to be posed incorrectly. By varying the prescribed velocities at the nozzle sections, one can ultimately choose a distribution of velocities that does not contradict the prescribed conditions at infinity. But there are infinitely many such solutions. In order to select a unique solution, it would be necessary to solve an analogous problem "behind the sections" of the nozzles, i.e., in the devices feeding the jets, and then to match ( "glue" ) the solutions along the nozzle sections.

Here a different formulation of the problem is proposed: placing the body being flowed around in the "physical" space and the jet devices in their own spaces, we cut these spaces along the nozzle sections and then glue them to one another along these sections in accordance with what their interaction is in the problem posed. Considering such an $n$-space and schematizing in one way or another the form of the body, the boundaries, and the jet devices, we now pose a single boundary-value problem (of the type of Dirichlet, Neumann, mixed problems, etc.) for the entire $n$-space. Such a formulation of the problem determines, generally speaking, a unique solution. We note that this approach to the problem may also prove useful in considering special cases of supersonic and mixed flows, when the supersonic flow at the nozzle section breaks down and becomes subsonic, as a result of which an interaction of the separate spaces arises. Naturally, the possibility of obtaining analytical solutions in the new formulation depends directly on the existence and development of a mathematical apparatus that makes it possible to study flows in an $n$-space, and also gas flows close to real ones. At present one can point to the theory of analytic functions, which makes it possible to consider the flow of an ideal fluid on an $n$-sheeted surface (exact solutions), or flows reducible to flows of an ideal fluid on an $n$-sheeted surface (for example, the application of the theory of S. A. Khristianovich to the construction of flow past bodies with jets in a compressible gas, the construction of a linear theory, the solution of various variational problems, etc.). Below we shall dwell only on two classes of exact solutions in an ideal fluid, assuming the flow to be plane, irrotational, steady, with one constant Bernoulli constant throughout the entire flow.

## II. Theory of smooth flow by an ideal fluid past an arbitrary plane body with jets.

Suppose that the body with jet devices is placed on an $n$-sheeted surface and (without loss of generality) that on each sheet there is only one infinitely remote point. We require that the $n$-sheeted body be simply connected

and call it an $n$-sheeted profile. We construct a conformal mapping of the exterior of the unit circle (the $\xi$-plane) onto the exterior of the $n$-sheeted profile (the Riemann surface $z$), under the condition that the infinitely remote point of the $\xi$-plane corresponds to the infinitely remote point of the first sheet of the Riemann surface (the "physical" plane), where

$$z(\xi) = \xi + a_0 + a_{-1}\xi + \dots + a_{-n}\xi^{-n} + \dots \tag{1}$$

Such a mapping is easy to construct for an arbitrary shape of an $n$-sheeted profile by applying, for example, the Christoffel-Schwarz formula, with account of (1), to an $n$-sheeted polygon inscribed in the $n$-sheeted profile, and then mapping the region close to the circle onto the circle. It is easy to see that the complex potential $F(\xi)$ of flow past an $n$-sheeted profile in the case of an unbounded flow in the "physical" plane will be

$$F(\xi) = v_\infty \left\{ \xi e^{-i\alpha} + \frac{e^{i\alpha}}{\xi} - \frac{1}{2\pi}\left( i\gamma + \sum_{k=2}^{n} q_k \right) \ln \xi + \frac{1}{\pi} \sum_{k=2}^{n} q_k \ln(\xi - \xi_k) \right\}, \tag{2}$$

where $\alpha$ is the angle of attack of the profile in the "physical"plane; $\Gamma = \gamma v_\infty$ is the circulation around the $n$-sheeted profile; $\xi_k = e^{i\vartheta_k}$ is the position of the infinitely remote points $A_k$ on the unit circle ($k > 1$); $q_k$ is the strength of the sources (or sinks) corresponding to $A_k$. We shall restrict ourselves to consideration of flows in which the critical points $\xi_{m,0}$, determined from the equation of degree $n+1$,

$$\frac{dF}{d\xi} = U_\infty \left\{ e^{-i\alpha} - \frac{e^{i\alpha}}{\xi^2} - \frac{1}{2\pi}\left( i\gamma + \sum_{k=2}^{n} q_k \right) \frac{1}{\xi} + \frac{1}{\pi} \sum_{k=2}^{n} \frac{q_k}{(\xi - \xi_k)} \right\} = 0, \tag{3}$$

lie on the unit circle ($\xi_{m,0} = e^{i\vartheta_{m,0}}$). We shall always assume that, for any $\alpha$ and $\{q_k\}$, on the $n$-sheeted profile under consideration there is an unchanged branch point $\xi_{1,0} = e^{i\vartheta_{1,0}}$, which we shall call the Joukowski-Chaplygin trailing edge. Then we obtain the following formula for determining the circulation:

$$\gamma = 4\pi\{\sin(\vartheta_{1,0} - \alpha) - \sin(\vartheta_{1,0} - \alpha_0)\}. \tag{4}$$

Here $\alpha_0$ is the angle of attack at zero circulation, which for an $n$-sheeted profile depends not only on its geometry but also on the set of discharges $\{q_k\}$, and is determined from the relation

$$\sum_{k=2}^{n} q_k \operatorname{ctg}\frac{1}{2}(\vartheta_{1,0} - \vartheta_k) - 4\pi\sin(\vartheta_{1,0} - \alpha_0) = 0. \tag{5}$$

Suppose that the body-jet-device system is completely adjusted for certain $\gamma = \gamma'$, $\alpha = \alpha'$, i.e., such discharges $q_k = q_k'$ and positions of the singular points $\xi_k$ have been selected that the branch points $\xi_{m,0}$ are located at prescribed points of the contour of the $n$-sheeted profile. It is of interest to consider such regulation of $\{q_k\}$ with respect to the angle of attack that, when $\alpha$ changes, the maximum possible number of branch points would preserve their prescribed position $\xi_{m,0}$. For this purpose one can compose $n-1$ equations for the compatibility of the existence, on the $n$-sheeted profile at one value of $\gamma$, of $n$ calculated branch points (including the point $\xi_{1,0}$):

$$\sum_{k=2}^{n} q_k \left[\operatorname{ctg}\frac{1}{2}(\vartheta_{m,0} - \vartheta_k) - \operatorname{ctg}(\vartheta_{1,0} - \vartheta_k)\right] = 4\pi[\sin(\vartheta_{m,0} - \alpha) - \\ - \sin(\vartheta_{1,0} - \alpha)]. \tag{6}$$

In this case there remains only one critical point, which we denote by $\xi_{n+1,0}$, which moves along the $n$-sheeted profile as the angle of attack changes and position of which $\xi_{n+1,0} = e^{i\vartheta_{n+1,0}}$ is determined from the relation

$$\vartheta_{n+1,0} - 2\alpha = \pi + \sum_{k=2}^{n} \vartheta_k - \sum_{m=1}^{n} \vartheta_{m,0}. \tag{7}$$

If branch points in the flow are not allowed, and if a change in the sign of $q_k$ is not allowed, then it is not difficult to see that the range of available variation of the angle of attack of the profile will be $|\Delta\alpha| \leq \frac{1}{2}|\vartheta_i - \vartheta_j|$, where $\vartheta_i$ and $\vartheta_j$ are points from the families

**Fig. 1**

Fig. 1

Figure 1: Fig. 1

$\{\xi_{m,0}\}$ and $\{\xi_k\}$, between which, for the computed value $\alpha = \alpha'$, the critical point $\xi_{n+1,0}$ is located. If not all $q_k$ are adjusted, but only $l$ of them, then the matching system (6) will consist of $l$ equations with $l+1$ critical points from $\{\xi_{m,0}\}$, which it is desirable to place at fixed points of the $n$-sheeted profile.

Define the force $R$ and the moment $M$ as the result of the action of the fluid on all rigid walls bounding the flow on all sheets of the Riemann surface. Suppose that the width of the jets at infinitely distant points ($k > 1$) is bounded and equal to $\delta_k$. Applying the momentum theorem to the flow on the $n$-sheeted surface, we shall have

$$\overline{R} = X - iY = \frac{\rho i}{2} \oint |F'(z)|^2 \, dz + \sum_{k=2}^{n} \delta_k \left[\rho v_k^2 \sin(\varphi_k - \vartheta_k) e^{-i\vartheta_k} + i p_k e^{-i\varphi_k}\right]. \quad (8)$$

Here $p_k, v_k, \vartheta_k$ are, respectively, the pressure, velocity, and angle of inclination of the velocity at the points $A_k$, and $\varphi_k$ is the inclination of the jet section along the normal at the points $A_k$ when the contour of the $n$-sheeted profile is traversed counterclockwise.

Accordingly, by the theorem of moments of momentum we write

$$M = \frac{1}{2}\rho \operatorname{Re} \oint |F'(z)|^2 z \, dz + \operatorname{Im} \sum_{k=2}^{n} \delta_k z_k \left[\rho v_k^2 \sin(\varphi_k - \vartheta_k) e^{-i\vartheta_k} + i p_k e^{-i\varphi_k}\right], \quad (9)$$

where $z_k$ is the complex distance along the normal from the moment point to the line of symmetry of the jets at the points $A_k$. Note that the first terms in (8) and (9) are the same as for a single-sheeted flow.

III. **Examples of exact solutions.** a) Let us construct the simplest smoothly streamlined mathematical model No. 1, which is the rectangle $EE'E_1'E_1$ with jet inlet $E_1E_1'$ and outlet nozzle section $EE'$ (Fig. 1). We attach to the sections $EE'$ and $E_1E_1'$ the simplest jet devices in the form of channels with walls parallel to the axis of symmetry of the apparatus, located respectively on the second and third sheets of the Riemann surface. For the simplest three-sheeted symmetric profile under consideration

$$z(\xi) = \xi + 1/\xi - 2\sin^2\vartheta_E \cdot \ln(\xi - 1)/(\xi + 1) + c, \quad (10)$$

where the angle $\vartheta_E$ is related to the elongation $\lambda$ of the model by

Fig. 2

Figure 2: Fig. 2

$$\lambda = (2/\pi)\,[\ln \operatorname{tg} \vartheta_E/2 + \cos\vartheta_E/\sin^2\vartheta_E]. \tag{11}$$

Let the Zhukovsky-Chaplygin point be $\xi_{1,0} = \xi_E = e^{i\vartheta_E}$. For angle of attack $\alpha \neq 0$, by means of (6) we match the positions of the critical points $\xi_{1,0} = \xi_E = e^{i\vartheta_E}$; $\xi_{2,0} = \xi_{E'} = e^{-i\vartheta_E}$ and $\xi_{3,0} = \xi_{E_1'} = e^{i\vartheta_E}$. Then the following must be carried out–

the following regulation of the discharges $q_2$ and $q_3$ is to be carried out:

$$\begin{aligned} q_2 &= 2\pi \sin\vartheta_E[\sin(\vartheta_E - \alpha) + \sin\alpha]; \\ q_3 &= -2\pi \sin\vartheta_E[\sin(\vartheta_E - \alpha) - \sin\alpha]. \end{aligned} \tag{12}$$

The position of the fourth branch point is $\vartheta_{4,0} = \pi - \vartheta_E + 2\alpha$, and the velocity of the flow at any point of the flow on the Riemann surface will be

$$dF/dz = U_\infty e^{-i\alpha}[\xi + e^{-i(\vartheta_E - 2\alpha)}]/[\xi + e^{-i\vartheta_E}]. \tag{13}$$

b) In the proposed formulation, in the presence of an infinite zone of constant pressure, one may directly make use of the methods developed in jet theory by Helmholtz, Kirchhoff, Zhukovsky, Chaplygin, Levi-Civita, Sedov, and many others. Let us construct mathematical model No. 2, which differs from model No. 1 by the presence of an additional jet with nozzle cut $DG$ and discharge $q_4$, entering from an infinitely distant point also through the simplest supply device (Fig. 2). Thus here we already have a flow on a four-sheeted Riemann surface. Suppose that the infinite zone of constant pressure is bounded by streamlines issuing from the points $G$ and $E'$. We shall require that the points $E$, $D$, and $E_1'$ be branch points. Following Zhukovsky, we obtain in the unbounded flow:

**Fig. 2**

$$F(t) = c_1 U_\infty \int \frac{(t - t_E)(t - t_{D_1})(t - t_{E_1})(t - t_D)}{(t - t_{A_2})(t - t_{A_3})(t - t_{A_1})}\, dt; \tag{14}$$

$$\ln \frac{U_\infty}{dF/dz} = ic_2 \int_\infty^t \frac{(t - t_{\mu_1})(t - t_{\mu_2})}{(t - t_{D_1})(t - t_{E_1})(t - t_D)} \frac{dt}{\sqrt{t(t-1)}}. \tag{15}$$

Here $t_B = 1$, $t_G = 0$, and $t_{A_1} = \infty$, while $t_{A_2}$, $t_{A_3}$, $t_{A_4}$ correspond to the infinitely distant points $A_2$, $A_3$, $A_4$.

Equations (14), (15) contain 12 unknown real quantities $(c_1, t_E, t_{E_1'}, t_D, t_{D_1}, t_{A_2}, t_{A_3}, t_{A_4}, c_2, t_{\mu_1}, t_{\mu_2}, t_{E_1})$. We are given the geometry of the four-sheeted body and the angle of attack of the flow at infinity, which makes it possible to compose the 12 necessary equations fixing: 1) the angle of attack; 2) the change in the angle of inclination of the velocity at the corner points $E_1$, $D$ and at the critical point $D_1$; 3) the length and angle of inclination of the cuts $EE'$, $E_1E_1'$, and $DG$; 4) the distance to the cut $DG$ ($l_{DE_1'}$); and 5) the characteristic length of the apparatus, which may be taken as unity.

Using the methods developed in jet theory, it is easy to write down general solutions under various boundary conditions (ground, free surface, rigid pipe, free jet, jet flowing from a pipe), and also when, along with an infinite zone of constant pressure, there is present a bounded zone of constant pressure $p_k < p_\infty$. Undoubtedly, only by computation can one determine precisely the domains of existence of the solutions noted in b) and the position of the extremal points, as well as the value and character of the regulation with respect to $\alpha$ of the discharges $q_2$, $q_3$, $q_4$.

Central Aerohydrodynamic Institute
named after N. E. Zhukovsky

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*Note: Figure translations are in progress. See original paper for figures.*