ON BOREL SETS IN PERFECTLY NORMAL BICOMPACTS

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ON BOREL SETS IN PERFECTLY NORMAL BICOMPACTS

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Perfectly normal bicompacts are a natural domain for constructing the (classical) theory of Borel sets: in these spaces the Borel sets constructed on the basis of open sets and on the basis of closed sets are one and the same. In the present note we shall construct Borel sets on the basis of open sets. The classification of Borel sets will be understood in the sense of Hausdorff (^{2}). The system K_0 consists of all open sets of the space X; the system K_1 is the collection of all sets of type G_δ. In general, for an arbitrary ordinal number $\alpha < \omega_1$, by K_α we denote the collection of all sets $M \subseteq X$, each of which is, for odd* (respectively even) α, the sum (respectively, intersection) of a countable number of sets $M_k \in K_{\alpha'}$, $\alpha' < \alpha$. We shall say that sets $M \in K_\alpha$ are obtained from open sets "according to scheme α." The class of a Borel set M is the least such α that $M \in K_\alpha$.

In this note the following is proved.

Main theorem ** (on nonemptiness of classes). *In an uncountable perfectly normal bicompact X, for every $\alpha < \omega_1$ there exist Borel sets of class α.*

The proof is based on the following propositions:

Lemma 1. *The perfect kernel of an uncountable perfectly normal bicompact X is nonempty, and consequently has the cardinality of the continuum* (^{1}).

Thus it suffices for us to prove our theorem for perfectly normal bicompacts without isolated points.

Lemma 2. *If X is a perfectly normal bicompact without isolated points, then there exists a continuous mapping $e : X \to I$ onto an interval of the number axis.*

This lemma was proved by A. Chernavskii in (^{3}), where, moreover, it is proved that in perfectly normal bicompacts without isolated points there exist A-sets that are not Borel sets.

Main lemma 3 **. Let $\sigma_0 = \{U_1, \ldots, U_i, \ldots\}$ be any countable system of open sets of a perfectly normal bicompact X. Then there exist a compact Y

and a continuous mapping $f : X \to Y$ under which each $U_i \in \sigma_0$ is marked, i.e. $f^{-1} f U_i = U_i$.*

Proof. Let $U_i \in \sigma_0$ be arbitrary. The set U_i has type F_σ, and the closed set $F_i = X \setminus U_i$ has type G_δ. Let

$$F_i = \bigcap_{j=1}^{\infty} \Gamma_{ij},$$

where the Γ_{ij} are open in X. Consider, for each j, an open cover ω_{ij} of the whole space X, consisting of only two elements: sets U_i and sets Γ_{ij}. For this cover ω_{ij} there exists a compactum Y_{ij} and a continuous ω_{ij}-mapping $f_{ij} : X \to Y_{ij}$.

Consider the continuous mapping $f_i : X \to Y_i \subseteq \prod_{j=1}^{\infty} Y_{ij}$ into the compactum $\prod_{j=1}^{\infty} Y_{ij}$, defined as follows:

$$f_i(x) = \{f_{ij} x\} \in \prod_{j=1}^{\infty} Y_{ij}.$$

This mapping is an ω_{ij}-mapping for every j (and fixed i). We shall prove that the set U_i is marked under the mapping f_i. Indeed, let $x_0 \in U_i$. It is necessary to prove that $f_i^{-1} f_i x_0 \subseteq U_i$. There exists a number $j = j_0$ such that $x_0 \in \Gamma_{ij_0}$; otherwise $x_0 \in \Gamma_{ij}$ for every j and, hence,

$$x_0 \in \bigcap_{j=1}^{\infty} \Gamma_{ij} = F_i,$$

whereas $x_0 \in U_i$. Thus $f_i^{-1} f_i x_0 \not\subset \Gamma_{ij_0}$. Further, $f_i : X \to Y_i \subseteq \prod_{j=1}^{\infty} Y_{ij}$ is an ω_{ij_0}-mapping. Therefore $f_i^{-1} f_i x_0$ is contained in some element of the cover ω_{ij_0}. But, since $f_i^{-1} f_i x_0 \not\subset \Gamma_{ij_0}$, necessarily $f_i^{-1} f_i x_0 \subseteq U_i$, which proves the markedness of the set $U_i \in \sigma_0$. Consider the continuous mapping

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* All limit transfinite numbers are even; a transfinite number $\alpha = \beta + n$, where β is a limit number and n is a natural number, has, by definition, the same parity as the number n.

** This theorem gives a positive solution to a problem posed by P. S. Aleksandrov in 1935.

*** This lemma was first proved in (^{6}).

$$f : X \to Y \subseteq \prod_{i=1}^{\infty} Y_i,$$

defined by the same rule:

$$fx = \{f_i x\} \in \prod_{i=1}^{\infty} Y_i.$$

Under the mapping f, every set $U_i \in \sigma_0$ will already be marked. The lemma is proved.

Main consequence. Let $\sigma_0 = \{U_1, \dots, U_i, \dots\}$ be an arbitrary countable system of open sets of the perfectly normal bicompactum X. Then there exists a compactum Y and a continuous mapping $f : X \to Y$ under which not only the sets $U_i \in \sigma_0$ are marked, but also every Borel set constructed on the basis of the system σ_0. Moreover, every A-set obtained on the basis of the system $\sigma_0 = \{U_1, \dots, U_i, \dots\}$ will be marked. Such a mapping is the mapping constructed in Lemma 3.

Proof of the main theorem. Let X be a perfectly normal bicompactum without isolated points. By Lemma 2, there exists a continuous mapping $e : X \to I$ onto a segment I of the numerical axis. Let $N \subseteq I$ be a Borel set in I of some class $\lambda \geq \omega_0$. It is sufficient to prove that the set $e^{-1}N = M$ is a Borel set in X of the same class λ. Denote the class of the set M by λ'. We must prove that $\lambda' = \lambda$.

a) We shall prove that $\lambda' \leq \lambda$. Since the class of the set N is λ, there exists a countable system $\sigma_0 = \{V_1, \dots, V_i, \dots\}$ of open subsets of I, from which the set N is obtained according to the scheme λ, and there does not exist a countable system of open subsets of I from which the set N would be obtained according to a scheme $\mu < \lambda$. Consider the system

$$e^{-1}\sigma_0 = \{e^{-1}V_1, \dots, e^{-1}V_i, \dots\}$$

of open subsets of X. Then the set $M = e^{-1}N$ is obtained from $e^{-1}\sigma_0$ according to the scheme λ, and there does not exist a countable system of open sets in X, marked under e, from which the set M would be obtained according to a scheme $\mu < \lambda$. Thus, $\lambda' \leq \lambda$.

b) Suppose now that $\lambda' < \lambda$. Then there exists a countable system $\sigma_1 = \{U_1, \dots, U_i, \dots\}$ of open subsets of X, from which our set $\cdots$

the set M is obtained according to the scheme λ'. As follows from the preceding, it is necessary that some sets of σ_1 not be marked under e.

By Lemma 3, there exists a compactum Y_1 and a continuous mapping $f_1 : X \to Y_1$ such that all sets $U_i \in \sigma_1$ are marked, and therefore, by the corollary to this lemma, every Borel set constructed on the basis of the system σ_1 will necessarily be marked under the mapping f_1; in particular, the set $M = e^{-1}N$ will be marked. Now consider the compactum $Y_1 \times I$ and the continuous mapping

$$f : X \to fX = Y \subseteq Y_1 \times I,$$

constructed as follows: $fx = \{f_1 x, ex\} \in Y_1 \times I$ for each point $x \in X$. Denote by π_{Y_1}, π_I the projections of the compactum $Y_1 \times I$ onto Y_1 and onto I, respectively. Then we obtain

$$ex = \pi_I fx; \qquad f_1 x = \pi_{Y_1} fx. \tag{*}$$

Denote by P_e, P_{f_1}, P_f the totality of all marked sets under the mappings e, f_1, f, respectively. We shall have

$$P_e \subseteq P_f; \qquad P_{f_1} \subseteq P_f. \tag{**}$$

Further, since the set $M = e^{-1}N$ is marked under the mapping $f_1 : X \to Y_1$, it is also marked under the mapping $f : X \to Y = fX$. Then, by virtue of $(*)$, we obtain

$$\pi_I f e^{-1} N = \pi_I f M = N.$$

Moreover, note that the set $fM = fe^{-1}N$ is marked under the mapping $\pi_I : Y \to I$. Consequently, under the mapping π_I the set fM is mapped completely onto the set N. Now consider the system

$$f\sigma_1 = \{fU_1, \dots, fU_i, \dots\}$$

of open sets in Y—open, since each $U_i \in \sigma_1$ is marked under the mapping f_1, and hence also under the mapping f. Then the set fM is obtained from the system $f\sigma_1 = \{fU_1, \dots, fU_i, \dots\}$ according to the scheme λ', and therefore the class of the Borel set fM in the compactum Y is equal to some $\lambda'' \leq \lambda' < \lambda$. Thus the absolutely Borel set $fM = fe^{-1}N$ of class $\lambda'' < \lambda$ under the mapping π_I, which maps it completely onto the absolutely Borel set $N \subseteq I$ of class $\lambda > \lambda''$, cannot exist, since λ is an infinite ordinal number, and the following assertion is true (see (5)).

Let f be a perfect mapping of an absolutely Borel set X of class α onto an absolutely Borel set Y. Then the class of the set Y is equal to the class of the set X, if $\alpha \geq \omega_0$, and the class of the set Y is finite, if $\alpha < \omega_0$.

Thus, in our bicompactum X there is a Borel set of any infinite class; consequently, there are also sets of any finite class. The theorem is proved.

In conclusion we shall give a simple proof of the following proposition, first proved by V. E. Shneider in (4).

Theorem 2. *Let* $X = X_1 \cup X_2$*, where* X *is a perfectly normal bicompactum, and* X_1 *and* X_2 *are two mutually complementary* A*-sets. Then* X_1 *and* X_2 *are necessarily Borel sets.*

The proof consists in an easy reduction to the same assertion for compacta, which constitutes the content of a well-known theorem of M. Ya. Suslin (see (2)). Indeed, by virtue of the corollary to Lemma 3, there exist com-

the compacta Y_1 and Y_2 and continuous mappings

$$f_1 : X \to Y_1; \qquad f_2 : X \to Y_2,$$

such that X_1 is marked* under f_1, and X_2 under f_2.

Consider the continuous mapping $f : X \to Y \subseteq Y_1 \times Y_2$, defined as follows: $fx = \{f_1x, f_2x\}$ for every $x \in X$. Then, under the mapping $f : X \to Y$, the set X_1 is marked, and the set X_2 is marked, and, moreover, fX_1 and fX_2 are necessarily A-sets in $Y = fX$. We shall have

$$Y = fX = fX_1 \cup fX_2; \qquad fX_1 \cap fX_2 = \Lambda.$$

It now remains only to apply Suslin' s theorem: the sets fX_1 and fX_2 are necessarily Borel. But then the set $X_1 = f^{-1}fX_1$ and the set $X_2 = f^{-1}fX_2$ are also Borel in the perfectly normal bicompactum X. The theorem is proved.

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* And, of course, marked are those open sets (of which there are countably many) from which X_i is obtained by applying the A-operation. Exactly the same applies to f_2.

Note: Figure translations are in progress. See original paper for figures.

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