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Abstract

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ON THE COMPUTATION OF SOUGHT SURFACES IN SPATIAL METHODS OF CHARACTERISTICS

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The numerical solution of most problems of supersonic steady gas flows is connected with the need to find previously unknown surfaces of the type of shock waves and free surfaces. For the computation of such surfaces in spatial methods of characteristics, it was proposed to use two compatibility conditions ([1–4]). In ([4]) an attempt was made to apply such a scheme in practice. In ([5]) it is noted that such a scheme, when computing the flow around bodies at an angle of attack, leads to incorrect results outside the region bounded by the body and the characteristic surface issuing from the outer boundary of the region where the initial data are prescribed, and a new method is proposed using one characteristic relation. In the present paper it is proved that, in difference schemes for finding and computing points of the sought boundary surfaces (a shock wave, a free surface), in addition to the finite conservation laws it is necessary and sufficient to use one combination of the differential equations of gas dynamics (for example, a compatibility condition along a certain bicharacteristic).

Let us take the equations of steady supersonic motion of an ideal gas in the form ([5])

$$\cos\gamma\mathbf{k}_2\nabla\beta + \mathbf{k}_3\nabla\gamma + Q_1\mathbf{k}_1\nabla p = F_1, \qquad \cos\gamma\mathbf{k}_1\nabla\beta + Q\mathbf{k}_2\nabla p = F_2,$$
$$\mathbf{k}_1\nabla\gamma + Q\mathbf{k}_3\nabla p = F_3, \qquad \mathbf{k}_1\nabla S = 0, \tag{1}$$

where the velocity vector in the cylindrical coordinate system z, r, φ is introduced according to the relation $\mathbf{V} = V\{\cos\beta\cos\gamma,\ \sin\beta\cos\gamma,\ \sin\gamma\}$; $\mathbf{k}_1 = \mathbf{V}/V$, $\mathbf{k}_2 = \cos^{-1}\gamma\,\partial\mathbf{k}_1/\partial\beta$, $\mathbf{k}_3 = \partial\mathbf{k}_1/\partial\gamma$ is a local Cartesian system; $F_i = F_i(\beta, \gamma, r)$, $Q = Q(P, S)$, $Q_1 = Q_1(P, S)$, and $\mathbf{k}_i\nabla f$ is the derivative in the direction of the vector $\mathbf{k}_i$.

Fig. 1

Figure 1: Fig. 1

Suppose that a numerical solution of equations (1) is sought in a region bounded on one of its portions by an unknown boundary and satisfying on the latter certain relations (boundary conditions).

Consider an elementary patch $BOCA$ of the constructed surface, which is one of the faces of a computational cell in a difference scheme (Fig. 1). Let the radius vector $\mathbf{r}$, the tangent vectors $\mathbf{r}'_s$ and $\mathbf{r}'_t$ in the directions OA and BC, respectively, be known at the points B, O, C. Assume also that the sought surface has a continuous and piecewise-smooth normal $\mathbf{n}$. We shall prove that, in order to find the radius vector $\mathbf{r}_A$ and the unit normal $\mathbf{n}_A$ at the computational point A of the sought surface, it is sufficient to invoke one combination of the gas-dynamic equations (1), for example, the characteristic relation along a definite wave characteristic.

Indeed, it is known from differential geometry that, when the corresponding conditions of continuity and smoothness are satisfied, $\mathbf{n}'_s = a\mathbf{r}'_s + b\mathbf{r}'_t$, or, in difference form,

$$\mathbf{n}_A = \mathbf{n}_O + (a\mathbf{r}'_s + b\mathbf{r}'_t)_O \Delta s + O(\Delta s^2), \tag{2}$$

where $(\ )'_s$ and $(\ )'_t$ are derivatives in the directions AO and BC, respectively; quantities at the point O are marked by the subscript O; and a and b are the unknown scalars. Using the relation $(\mathbf{r}'_t\mathbf{n})'_s = 0$, we obtain the condition relating a and b:

$$ar'_s r'_t + br^2_{t'} = -nr''_{ts}. \tag{3}$$

In this relation $\mathbf{r}'_s$, $\mathbf{r}'_t$, and $\mathbf{n}$ at the point O are known, while the right-hand side can be computed by a difference relation from the data at the points B and C, taking into account the equality $r''_{ts} = r''_{st}$.

The radius vector of the point is determined by the formula

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r}'_{sO}\Delta s + O(\Delta s^2). \tag{4}$$

Fig. 1

Thus, prescribing the radius vectors and normals at the points B, C, and O determines a nearby point A of the desired surface by (4) and the unit normal at it by (2), except for one scalar quantity, for example a, in (3). To find this quantity one must invoke some combination of equations (1).

Fig. 2

Figure 2: Fig. 2

In the proof, continuity was used inside the region $ABOC$ of the geometric parameters up to the first derivatives of the normal. This restriction is not burdensome in numerical computation, since continuity can be violated only along a finite number of lines on the surface (for example, lines of influence of breaks and discontinuities of the curvature of a body on a shock wave), and the computational scheme can be chosen so that the line of possible discontinuities of the derivatives passes along the boundary BOC or through the point A.

Relations (2)–(4) are obvious geometric properties of the surface and, in any scheme of the spatial method of characteristics (more precisely, in any numerical method), when computing an arbitrary boundary surface, must either be used directly or their fulfillment must be ensured.

Further, throughout we shall consider the equation of the desired surface $\mathbf{r} = R(z, \varphi)$ and its unit normal in the form

$$\mathbf{n} = (1 + q^2 + \omega^2)^{-1/2}\{q, -1, \omega\}, \qquad (q = \partial R/\partial z,\ \omega = R^{-1}\partial R/\partial \varphi). \quad (5)$$

Fig. 2

Let us consider some particular examples. If a known boundary surface is considered, for example a body being flowed around, then the fulfillment of conditions (2), (4) is obvious, since (5) is obtained by differentiating the function $R(z, \varphi)$. In the case of implicit difference schemes (6), in particular semicharacteristic methods of type (7), the component of the normal at the point A (Fig. 2) is computed by formulas of the type

$$\omega_A = (R(A'') - R(A'))/2R\Delta\varphi + O(\Delta\varphi^2), \quad (6)$$

and the unknown will remain q_A. In this case the equality of the second mixed derivatives is satisfied in difference form,

$$\partial^2 R/\partial z\,\partial\varphi = (R(A'') - R(A') - R(C) + R(B))/2\Delta z\Delta\varphi + O(\Delta z^2 + \Delta\varphi^2) = \partial^2 R/\partial\varphi\,\partial z.$$

For axisymmetric (plane) surfaces (lines) $\omega = 0$, and the unknown at the point A remains q.

We shall show that, in constructing difference schemes for finding and computing points of a free surface and a shock wave, one cannot invoke more than one combination of equations (1) containing derivatives derived from the surface.

From this will follow the impossibility of the use, proposed in works (1–4), of two characteristic relations.

Let $BOCA$ (Fig. 1) be an element of the sought free surface, on which the boundary conditions $p = p_0 = \text{const}$ and $\mathbf{V} \cdot \mathbf{n} = 0$ must be satisfied. In computing the geometric and gasdynamic functions at the point A, it is necessary to find six unknowns: $\beta, \gamma, p, S, q, \omega$. At first sight it would seem that, to determine them, we have six equations: equations (1) and two boundary conditions. It turns out, however, that the difference approximations of equations (1) and of the boundary conditions do not determine all six unknowns uniquely at the computational point A. Indeed, suppose that on the plane P (Fig. 1), containing the vectors $\mathbf{k}_2$ and $\mathbf{k}_3$, initial data are prescribed. According to the boundary condition, along the free surface $\mathbf{k}_1 \nabla p = 0$, and the unknowns at the point A will not enter the first equation of system (1) in difference form. From the remaining equations (1) one can determine β, γ, S, from the boundary conditions p, while for determining the two geometric quantities q and ω there is only one condition, $\mathbf{V} \cdot \mathbf{n} = 0$. The missing condition is supplied by a relation of type (6) in implicit schemes (^{6,7}), or by (3) in explicit ones (^{5}).

Let us now consider a computational point A of a shock wave. From the conservation laws for passage through a shock discontinuity (^{5}), the normal components q and ω, as well as the pressure and entropy, can be expressed in terms of β and γ:

$$q = q(\beta, \gamma), \qquad \omega = \omega(\beta, \gamma); \tag{7}$$

$$p = p(\beta, \gamma), \qquad S = S(p) = S(\beta, \gamma). \tag{8}$$

Relations (7) express the geometric quantities in terms of the gasdynamic parameters β and γ. By formal calculations one can verify that the Jacobian of the transformation (7) is nonzero ($q_\beta \omega_\gamma - q_\gamma \omega_\beta \neq 0$), and also $\omega_\gamma \neq 0$. Relations (8) (the first of them is an analogue of the two-dimensional shock polar and the second is the Hugoniot condition) are boundary conditions for equations (1) and can be expressed in differential form:

$$p_\beta \, d\beta/ds + p_\gamma \, d\gamma/ds - dp/ds = 0, \qquad S_p \, dp/ds - dS/ds = 0 \; (S_p \neq 0). \tag{9}$$

Here df/ds are derivatives, generally speaking, in any direction on the surface.

It turns out that, in the three-dimensional case, in addition to (9), one more differential relation must be satisfied between the parameters on the shock wave. Differentiating in (7) the first relation with respect to φ, and the second with respect to z, and taking into account the identity $q_\varphi = \partial^2 R/\partial z \, \partial \varphi = \partial(R\omega)/\partial z = q\omega + R\partial\omega/\partial z$, we obtain:

$$\omega_\beta \frac{\partial \beta}{\partial z} + \omega_\gamma \frac{\partial \gamma}{\partial z} = \frac{1}{R}\left(q_\beta \frac{\partial \beta}{\partial \varphi} + q_\gamma \frac{\partial \gamma}{\partial \varphi}\right) - \frac{\omega}{R} \qquad (\omega_\gamma \neq 0). \tag{10}$$

This equation is satisfied identically in the axisymmetric or plane case ($\omega = \gamma = \partial\beta/\partial\varphi = 0$). The analogue of the boundary condition (10) in implicit difference schemes is relation (6).

Let initial data be known in the region $BOCD$ of a plane of spatial type (Fig. 1). Construct a solution at the nearby sought point A of the shock wave passing through the line BOC. We shall prove that in doing so one cannot use more than one linear combination of the gasdynamic equations (1).

Write equations (1) in such a way that they include derivatives in the directions $s(OA)$, $l(OD)$, $t(BC)$ (Fig. 1), and solve them with respect to the derivatives with respect to s (this is possible, since the plane P is not characteristic):

$$a_{ij} du_j/ds = \Phi_i \qquad (\Phi_i = b_{ij} du_j/dl + c_{ij} du_j/dt + d_i), \tag{11}$$

where $i = 1, 2, \ldots, 4$, $u_1 = \beta$, $u_2 = \gamma$, $u_3 = p$, $u_4 = S$, and summation over the repeated index j is performed from 1 to 4.

Similarly, we rewrite equation (10):

$$\omega_\beta d\beta/ds + \omega_\gamma d\gamma/ds = \Phi_0 \quad (\Phi_0 = c_{01} d\beta/dt + c_{02} d\gamma/dt + d_0 \neq 0). \tag{12}$$

Consider equations (9), (12), (11) in a neighborhood of the point O. We compute the coefficients of the derivatives and the free terms of these equations with respect to the parameters at the point O, and the derivatives on the right-hand side from the data in the region P. The parameters at the point A can be found if du_j/ds is known:

$$u_{jA} = u_{j0} + (du_j/ds)_0 \Delta s + O(\Delta s^2), \tag{13}$$

then, from (7), q and ω can be determined at the point A.

Thus, for the four unknowns $d\beta/ds$, $d\gamma/ds$, dp/ds, dS/ds we have seven equations (9), (12), (11). The choice of four equations from them must satisfy the following conditions: a) the chosen conditions are linearly independent, and b) any of equations (9), (12) that follow from the boundary conditions is either used directly or is a linear combination of the remaining equations. The first condition is obvious. If the second condition is not satisfied, the solution at the point A does not satisfy all the boundary conditions.

Equations (9), (12) are independent of one another; for example, the third-order minors in the middle and on the right of the corresponding matrix, equal respectively to $\omega_\gamma(1 - S_p)$ and $\Phi_0(1 - S_p)$, are nonzero. We shall show that (12) cannot be obtained as a linear combination of equations (11). The same can be proved similarly for (9). Suppose the contrary. Let (12) be a consequence of equations (11). Then, in particular, the equality

$$\Phi_0 = a_i \Phi_i \quad (a_1^2 + a_2^2 + a_3^2 + a_4^2 \neq 0)$$

is valid. Since the left-hand side of this equality contains no derivatives with respect to l (whereas they depend not only on the data along BOC), the coefficients of du_j/dl on the right must be equal to zero, i.e. $b_{ij}a_i = 0$ $(j = 1, \dots, 4)$. But this means that on the shock wave some combination of the gas-dynamic equations (1) is satisfied that contains derivatives in two directions $s(OA)$ and $t(BC)$, i.e. the shock wave is a characteristic surface, which is false. Consequently, the solution obtained using two combinations (11) will differ from the solution using (12) by quantities of order $(u_j(D) - u_j(O))\Delta s/\Delta l$, i.e., when $O_l(\Delta s) = O(\Delta l)$, by a quantity $O(\Delta s)$, whereas, according to (13), this difference should be of order $O(\Delta s^2)$.

It follows from this that, when using two characteristic relations (two combinations of equations (1), or (11)), equation (12) is not satisfied.

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