HOMOLOGIES OF SPACES OF SMOOTH EMBEDDINGS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

513.836

MATHEMATICS

A. S. SHVARTS

HOMOLOGIES OF SPACES OF SMOOTH EMBEDDINGS

(Presented by Academician P. S. Aleksandrov, 15 VI 1965)

In the present note we outline proofs of some theorems that make it possible to obtain information on the homologies of the space of smooth embeddings of a smooth manifold V^k into a smooth manifold W^n in the case when $n > 2k+1$.* As an application of these theorems, the homology groups of the space of embeddings of the circle S^1 in the sphere S^n are computed in dimensions $\leq 3n-10$.

By V^k and W^n we shall henceforth denote smooth manifolds of dimensions k and n, respectively; the manifold V^k will be assumed compact. By $Pl(V^k, W^n)$ we denote the space of smooth embeddings of V^k in W^n, endowed with the C^r-topology,** and by $Pl_0(V^k, W^n)$, or simply Pl_0, the space of based embeddings, i.e. the subspace of the space Pl consisting of embeddings that carry a marked point of V^k to a marked point of W^n and have at this point the prescribed differential.

The manifolds V^k and W^n with small open balls, correctly situated in them, removed will be denoted by V_1^k and W_1^n; the space of immersions of V_1^k into W_1^n that carry the boundary of the manifold V_1^k, by means of a standard diffeomorphism, into a certain sphere S^{k-1} lying on the boundary of the manifold W_1^n, in such a way that the image of V_1^k does not meet the boundary of W_1^n, will be denoted by $Im(V^k, W^n)$, or simply Im. (It is easy to see that Pl_0 is homotopy equivalent to the set $A_0 \subset Im$, consisting of embeddings (see (1)).)

We shall also use the notation $Im_1^r = Im_1^r(V^k, W^n)$ for the subset of $Im(V^k, W^n)$ consisting of immersions with one multiple point of multiplicity r; A_1 for the subset of Im_1^2 consisting of immersions for which the tangent planes to the image of V_1^k at the double point are in general position.

Proposition 1. *If $n \geq 2k+1$, then in dimensions $i \leq 2n-4k-2$ there is an exact sequence*

$$H_{i+1-n+2k}(A_1; Z_2) \to H_i(Pl_0; Z_2) \to H_i(Im; Z_2) \to H_{i-n+2k}(A_1; Z_2).$$

For the proof of this assertion we first note that the spaces $A_0 \cup A_1$ and A_1 may be regarded as infinite-dimensional smooth manifolds, with A_1 being a submanifold of codimension $n-2k$ in the manifold $A_0 \cup A_1$ and closed in $A_0 \cup A_1$. From this fact follows the relation

$$H_i(A_0 \cup A_1, A_0; Z_2) = H_{i-n+2k}(A_1; Z_2)$$

(see, for example, (2)). Further, we note that

$$H_i(A_0 \cup A_1; Z_2) = H_i(Im; Z_2)$$

for $i \leq 2n-4k-2$ (this follows from the fact that the manifold $A_0 \cup A_1$ is obtained from the manifold Im by deleting a submanifold having, in a certain sense, codimension $2n-4k$; the careful proof is easiest to carry out by passing to a finite-dimensional approximation). The required the assertion is obtained if one considers the exact sequence of the pair $(A_0 \cup A_1, A_0)$ and uses the relations indicated above.

Remark 1. It is convenient to study the homology of the space A_1 by means of the fibration obtained by assigning to each immersion f from A_1 the pair of points of the manifold V_1^k that are "glued together" under the immersion f. The base of this fibration is the space V_2 of unordered pairs of distinct points of the manifold V_1^k; its fiber, in turn, may be regarded as the space of a fibration whose base is the space of $2k$-frames tangent to the manifold W_1^n, while the fiber in dimensions $\leq n-2k-2$ (i.e., in almost all dimensions of interest to us) has the same homology groups as the space of a triad of based immersions of the manifold V^k into the manifold W^n (i.e., immersions taking three marked points of V^k to three marked points of the manifold W^n and having prescribed differentials at these points).

Remark 2. The exact sequence of Proposition 1 also holds for any coefficient group G, if one regards the homology groups of the space A_1 as being taken with respect to the corresponding, suitably chosen local coefficient system $\{G\}$. Using the considerations of Remark 1, one can compute the fundamental group of the space A_1; in particular, if $n > 2k+1$ and W^n is a simply connected manifold, then $\pi_1(A_1) = \pi_1(V_2)$.

Consider in the group $\pi_1(V_2)$ the image R of the fundamental group of the space $\tilde{V}_2$ of ordered pairs of distinct points of the manifold V_1^k under the natural map $\tilde{V}_2 \to V_2$. The action of the group $\pi_1(A_1)$ on the group G, which produces

---

* By the word smooth we mean differentiable a sufficiently large number of times.

** The number r is assumed sufficiently large; the homotopy properties of the spaces $Pl(V^k, W^n)$ and $Pl_0(V^k, W^n)$ do not depend on the choice of the number $r \geq 1$.

the local coefficient system we need, is described as follows: to the elements of the subgroup $R \subset \pi_1(V_2) = \pi_1(A_1)$ one assigns the identity transformation of the group G, and to the remaining elements the automorphism $\alpha(g) = -g$. In particular, if the manifold V^k is homeomorphic to the circle S^1, then the local coefficient system G is trivial (this is easily seen directly).

Let us note that Proposition 1 immediately implies the main result of the note by A. M. Vinogradov (1): the space $Pl_0(S^1, S^n)$ is acyclic in dimensions $\leq 2n-7$. Indeed, in this case Im is homotopy equivalent to $\Omega(S^{n-1})$, i.e. $H_i(Im; \mathbb{Z}) = 0$ for $i \neq k(n-2)$, $H_{k(n-2)}(Im; \mathbb{Z}) = \mathbb{Z}$. It is also easy to verify that A_1 is acyclic in dimensions $< n-3$; therefore, in order to compute the groups $H_i(Pl_0; \mathbb{Z})$ for $i \leq 2n-7$, it is necessary only to study the homomorphism $H_{n-2}(Im; \mathbb{Z}) \to H_0(A_1; \mathbb{Z})$ occurring in the exact sequence of Proposition 1. By a direct geometric construction it is easily shown that this homomorphism is an isomorphism.

Proposition 1 can also be used to obtain the following more general result.

Proposition 2. *The space $Pl_0(S^k, S^n)$ of based embeddings of the sphere S^k in the sphere S^n is acyclic in dimensions $< 2n-4k-2$.*

However, a simpler proof of this assertion will be sketched below on the basis of somewhat different considerations.

One can construct a spectral sequence containing within it the exact sequence of Proposition 1 and making it possible to obtain information about the homology groups of the space Pl_0 also in dimensions higher than $2n-4k-2$. For simplicity, we shall formulate the corresponding assertion only for spaces of embeddings of the circle and for dimensions $\leq 3n-8$.

Thus, let $V^k = S^1$; let A_0 and A_1 be the sets constructed above; let A_2 be the subset of $Im(S^1, W^n)$ consisting of immersions with two multiple points of multiplicity 2, at each of which the tangents do not coincide; let A_3 be the subset Im_1^2, consisting of immersions having first-order contact at the multiple point; let A_4 be the subset Im_1^3, characterizing-

···by the fact that, for the immersions entering it, the tangent lines at the multiple point are in general position.

Proposition 3. There exists a spectral sequence $\{E_r^{p,q}\}$, in which

$$E_1^{0,q} = H_q(Pl_0(S^1, W^n); Z), \quad E_1^{1,q} = H_{q-n+3}(A_1; Z),$$

$$E_1^{2,q} = H_{q-2n+6}(A_2; Z), \quad E_1^{3,q} = H_{q-2n+6}(A_3 \cup A_4; Z), \quad E_1^{p,q} = 0$$

for $p \geq 4$ and $p < 0$, and for $l \leq 3n-8$ the group

$$\sum_{p+q=l} E_\infty^{p,q}$$

is a group associated with the group $H_l(Im(S^1, W^n); Z)$ in some filtration.

The proof is based on the following considerations. Consider in the space $Im(S^1, W^n)$ the subspace

$$E = \bigcup_{0 \leq i \leq 4} A_i$$

and in this subspace the filtration $E_0 = A_0,\ E_1 = A_0 \cup A_1,\ E_2 = A_0 \cup A_1 \cup A_2,\ E_3 = E$. As is known, with the aid of such a filtration one can construct a spectral sequence for which

$$E_1^{p,q} = H_{p+q}(E_p, E_{p-1}; Z),$$

and

$$\sum_{p+q=l} E_\infty^{p,q}$$

is a group associated with the group $H_l(E; Z)$ endowed with a certain filtration. This is the spectral sequence we need, since

$$H_l(E; Z) = H_l(Im; Z)$$

for $l \leq 3n-8$ (because $Im \setminus E$ has, in E, in a certain sense codimension $3n-6$), and the homology groups $H_q(E_p, E_{p-1}; Z)$ are expressed in terms of the homology groups $H_s(E_p \setminus E_{p-1}; Z)$, in view of the fact that E_p may be regarded as an infinite-dimensional smooth manifold, and $E_p \setminus E_{p-1}$ as a closed submanifold of the manifold E_p. (The local system of coefficients on $E_p \setminus E_{p-1}$ turns out to be trivial.)

Proposition 4. The space $Pl_0(S^1, S^n)$ for $n \geq 4$ is acyclic in all dimensions $\leq 3n-10$, with the exception of dimension $2n-6$; the group

$$H_{2n-6}(Pl_0(S^1, S^n); Z)$$

is a free cyclic group.

For the proof, note that, in the dimensions of interest to us, in the spectral sequence of Proposition 3 among the groups $E_1^{p,q}$ with $p \geq 1$ there are only the following nonzero groups:

$$E_1^{1,n-3} = Z, \quad E_1^{1,2n-6} = Z, \quad E_1^{1,2n-5} = Z, \quad E_1^{2,2n-6} = Z+Z+Z, \quad E_1^{3,2n-6} = Z+Z+Z$$

(the difficulty in establishing these relations consists only in computing the groups $E_1^{1,q} = H_{q-n+3}(A_1; Z)$; however, since we are interested only in the homology groups $H_i(A_1; Z)$ in dimensions $\leq 2n-7$, this difficulty can be overcome by the methods used in the proof of Proposition 1).

Next, knowing the homology groups

$$H_l(Im(S^1, S^n); Z) = H_l(\Omega(S^{n-1}); Z),$$

we obtain information about the groups $E_\infty^{p,q}$; namely,

$$E_\infty^{p,q} = 0 \quad \text{for } p+q \neq k(n-2),$$

and the groups

$$\sum_{p+q=n-2} E_\infty^{p,q} \quad \text{and} \quad \sum_{p+q=2n-4} E_\infty^{p,q}$$

are groups associated with the group Z in some filtration. By direct geometric considerations one can compute the differential d_1 on the groups $E_1^{3,2n-6}$, $E_1^{1,n-3}$, and $E_1^{1,2n-5}$; it turns out that

$$d_1 E_1^{3,2n-6} = E_1^{2,2n-6}, \quad d_1 E_1^{1,n-3} = 0, \quad d_1 E_1^{1,2n-5} = 0.$$

From the relation

$$d_1 E_1^{3,2n-6} = E_1^{2,2n-6}$$

it follows, obviously, that

$$d_1 E_1^{2,2n-6} = 0;$$

noting in addition that

$$E_2^{1,2n-6} = E_\infty^{1,2n-6} = 0, \quad E_2^{0,2n-6} = E_1^{0,2n-6},$$

we see that the homomorphism

$$d_1 : E_1^{1,2n-6} \to E_1^{0,2n-6}$$

is an isomorphism. Thus

$$H_{2n-6}(Pl_0) = E_1^{1,2n-6} = Z.$$

From the structure of the differential d_1 described above it is clear that, among the groups $E_2^{p,q}$ with $p > 0$, in the dimensions of interest to us only the groups

$$E_2^{1,n-3} = Z \quad \text{and} \quad E_2^{1,2n-5} = Z$$

are different from zero; on the other hand, evidently,

$$E_1^{0,q} = E_2^{0,q} \quad \text{for } q \neq 2n-6.$$

By dimensional considerations it is clear that

$$E_2^{p,q} = E_\infty^{p,q} \quad \text{for } p+q \leq 3n-10.$$

The structure of E_∞ known to us gives the exact sequence

$$0 \to E_2^{0,q} \to H_q(Im) \to E_2^{1,q} \to 0,$$

from which it follows that

$$E_2^{0,q} = 0$$

(under the homomorphism Z to Z the kernel is equal to 0). Thus, for

$$q \leq 3n-10, \quad q \neq 2n-6,$$

$$H_q(Pl_0) = E_1^{0,q} = E_2^{0,q} = 0.$$

Proposition 5. *If* W^n *is a compact simply connected manifold of dimension* > 3*, then all homology groups* $H_i(Pl_0(S^1, W^n); \mathbb{Z})$ *are of finite type.*

This assertion is of some interest, since the group $H_0(Pl_0(S^1, S^3); \mathbb{Z})$ is not of finite type (there exists an infinite set of non-isotopic knots). The proof is based on Proposition 3 (more precisely, on its generalization, in which, by considering spaces of immersions with all possible types of singularities, the dimensional restriction $l \leqslant 3n - 8$ is removed).

The computation of the homology groups of the space $Pl_0(V^k, S^n)$ can also be carried out by means of a spectral sequence relating the homology groups of $Pl_0(V^k, S^n)$ to the homology groups of $Pl_0(V^k, S^{n+1})$. We shall now describe this spectral sequence in small dimensions. First note that the space S_1^n is homeomorphic to the ball E^n; therefore there is a natural projection $S_1^{n+1} \to S_1^n$, inducing a mapping λ of the space $Pl_0(V^k, S^{n+1})$ into the space of mappings $V_1^k \to S_1^n = E^n$. Using this, we distinguish in the space $Pl_0(V^k, S^{n+1})$ the subspaces $B_0 = \lambda^{-1}(A_0)$, $B_1 = \lambda^{-1}(A_1)$, $B_2 = \lambda^{-1}(R)$, where R is the set of mappings $V_1^k \to S^n$ having no multiple points and having differential of rank k at all points except one, where the rank of the differential is $k - 1$, and in the direction of degeneration of the first differential the second differential does not degenerate. It is easy to see that B_0 is homotopy equivalent to A_0, i.e. to $Pl_0(V^k, S^n)$; B_1 is homotopy equivalent to a double covering over A_1, and B_2 to a double covering over $\lambda(B_2)$.

Considering the spectral sequence associated with the filtration $E_0 = B_0 \subset E_1 = B_0 \cup B_1 \subset E_2 = B_0 \cup B_1 \cup B_2$, we obtain, by arguments entirely analogous to those used in the proof of Proposition 3, the following assertion.

Proposition 6. *If* $n \geqslant 2k + 1$*, then there exists a spectral sequence for which*

$$E_1^{0,q} = H_q(Pl_0(V^k, S^n); \mathbb{Z}), \qquad E_1^{1,q} = H_{q-n+2k+1}(B_1; \mathbb{Z}),$$

$$E_1^{2,q} = H_{q-n+2k+1}(B_2; \mathbb{Z}), \qquad E_1^{p,q} = 0 \quad \text{for } p \geqslant 3 \text{ and } p < 0,$$

and for $l \leqslant 2n - 4k - 2$ *the group*

$$\sum_{p+q=l} E_\infty^{p,q}$$

is a group associated with $H_l(Pl_0(V^k, S^{n+1}); \mathbb{Z})$ *for some filtration.*

With the help of this proposition a simple proof of Proposition 2 can be given. Namely, in the spectral sequence of Proposition 5 for $V^k = S^k$ the groups $E_1^{p,q} = 0$ for $p > 0$, $q \leqslant 2n - 4k - 3$, with the exception of the groups

$$E_1^{1,n-2k-1} = E_2^{2,n-2k-1} = E_1^{1,n-k-2} = E_1^{2,n-k-2} = \mathbb{Z}.$$

Geometric arguments show that

$$d_1 E_1^{2,n-2k-1} = E_1^{1,n-2k-1},$$

$$d_1 E_1^{2,n-k-2} = E_1^{1,n-k-2};$$

therefore $E_2^{p,q} = 0$ for all $p > 0$, $q \leqslant 2n - 4k - 3$. Hence we conclude that

$$H_i(Pl_0(S^k, S^n); \mathbb{Z}) = H_i(Pl_0(S^k, S^{n+1}); \mathbb{Z})$$

for $i \leqslant 2n - 4k - 3$, i.e.

$$H_i(Pl_0(S^k, S^n); \mathbb{Z}) = H_i(Pl_0(S^k, S^r); \mathbb{Z})$$

for $i \leqslant 2n - 4k - 3$ and $r > n$. Since, in an obvious way,

$$H_i(Pl_0(S^k, S^r); \mathbb{Z}) = 0$$

for sufficiently large r (see, for example, Proposition 1), Proposition 2 follows.

Moscow Engineering Physics Institute

Received
11 VI 1965

REFERENCES

1. A. M. Vinogradov, DAN, **156**, No. 5 (1964).

2. J. Eells, Proc. Nat. Acad. Sci. U.S.A., **45**, 1951 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.