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Abstract

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MATHEMATICS

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ON EXPANSION IN A SERIES IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEM

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1°. On the interval $[0, a]$ the boundary-value problem

$$Ly = -y'' + q(x)y = s^2 y, \tag{1'}$$

$$y(0) = 0, \qquad y'(a) + isy(a) = 0, \tag{1''}$$

is considered, where $q(x)$ is a complex-valued function, and s is the spectral parameter. Problem (1) arises in various questions of mathematical physics. Redheffer, who studied it in connection with scattering theory, showed that if $q(x)$ in a left half-neighborhood of the point a satisfies the condition

$$q(x) \sim c_\mu (a-x)^\mu, \quad x \to a-0; \qquad \mu \geqslant 0, \quad c_\mu \neq 0, \tag{2}$$

then the problem has a discrete spectrum s_n, and the system of eigenfunctions

$$\varphi_n(x), \qquad n = \pm 1,\ \pm 2, \dots, \tag{3}$$

is complete in $L_2(0, a)$*.

Let us point out a remarkable feature of problem (1): for $q(x) \equiv 0$ the spectrum is absent altogether.

From the completeness, established by Redheffer, of the system (3), generally speaking, the convergence of an expansion in a series with respect to this system does not follow. The present article is devoted to the study of this question.

Note that the expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(x), \quad x \in [0, a),$$

whose convergence is established below, is the expansion of the integral

$$f(x) = \int_0^{\infty} F(\lambda)\varphi(x, \lambda)\, d\rho(\lambda), \qquad 0 \leqslant x < \infty,$$

generated by the operator $Ly = -y'' + q(x)y$ in $L_2(0, +\infty)$ with finite potential $q(x)$, into a series by residues and, consequently, makes it possible to replace the Fourier integral by a series in the system of solutions (3), which in a number of cases turns out to be very convenient.

2°. Let us formulate the results obtained by us.

Theorem 1. *Let in (2) μ be equal to 0 or 1, and let $f(x)$ be twice differentiable on $[0, a]$, $f(0) = 0$, and for $\mu = 1$ also $f'(a) = 0$. Then $f(x)$ is expanded into the series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(x) =$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \varphi_n(x) \left[\int_0^a f(t)\varphi_n(t)\, dt - i f(a) \frac{\varphi_n(a)}{s_n} \right] \Big/ \left[\int_0^a \varphi_n^2(t)\, dt - i \frac{\varphi_n^2(a)}{2 s_n} \right], \tag{4}$$

* In ([1]) it is proved, moreover, that if the functions (3) are continued smoothly to the interval $[a, 2a]$ by the f

converging everywhere on the half-open interval $[0, a)$ and uniformly inside this interval. Moreover,

$$\left| f(x) - \sum_{n=-N_1}^{N_2} c_n \varphi_n(x) \right| \leq \text{const} \cdot N^{-\delta/b}$$

uniformly in $x \in [0, a - \delta]$, where $b = 2a/\mu + 2$, and $N = \min(N_1, N_2)$.*

For the proof we use the method of contour integration (see ([3])). Denote by $y_1(x, s)$ and $y_2(x, s)$ the solutions of equation (1′) satisfying the conditions

$$y_1(0, s) = 0, \qquad y_1'(0, s) = 1; \qquad y_2(a, s) = 1, \qquad y_2'(a, s) = -is,$$

and by $\omega(s) = W(y_1, y_2)$ the Wronskian determinant of these solutions:

$$\omega(s) = -y_1'(a, s) - isy_1(as) = -y_2(0, s).$$

It can be shown that, as $s \to \infty$, $s = \sigma + i\tau$, the following formulas hold:

$$y_1(x, s) = \frac{\sin sx}{s} + O\left(\frac{e^{|\tau|x}}{s^2}\right), \tag{5}$$

$$y_2(x, s) = e^{is(a-x)}\left[1 + O\left(\frac{1}{s}\right)\right] + \frac{c_\mu \Gamma(\mu+1)}{(-2is)^{\mu+2}} e^{-is(a-x)}\left[1 + O\left(\frac{1}{s}\right)\right], \tag{6}$$

$$\omega(s) = -y_2(0, s) = -e^{isa}\left[1 + O\left(\frac{1}{s}\right)\right]\left[1 + O\left(\frac{1}{s}\right) - \frac{e^{-2isa}}{As^{\mu+2}}\right], \tag{7}$$

where $A = -(-2i)^{\mu+2}/c_\mu \Gamma(\mu+1)$, so that the eigenvalues s_n asymptotically satisfy the equation

$$e^{-2isa} = As^{\mu+2}\left[1 + O(1/s)\right].$$

Solving it, we obtain for $s_n = \sigma_n + i\tau_n$

$$\sigma_n = \frac{n\pi}{a} - \frac{1}{2a}\arg A + O\left(\frac{\ln n}{n}\right),$$

$$\tau_n = \frac{\mu+2}{2a}\ln|n| + \frac{1}{2a}\ln\left[\left(\frac{\pi}{a}\right)^{\mu+2}|A|\right] + O\left(\frac{1}{n}\right). \tag{8}$$

From these formulas it is evident, in particular, that, beginning with some n, all eigenvalues of problem (1) are simple.

The principal asymptotic terms of formulas (8) were obtained by a different method by Reddhe ([2]). The method we use makes it possible to obtain for s_n any number of terms of the asymptotic expansion.

After this, consider, as usual, the nonhomogeneous equation $Ly - s^2 y = f(x)$. Its solution $\Phi(x, s)$, satisfying the conditions $(1'')$, has the form

$$\Phi(x, s) = \int_0^a G(x, t; s) f(t)\, dt =$$

$$= \frac{y_2(x, s)}{-\omega(s)}\int_0^x y_1(t, s) f(t)\, dt + \frac{y_1(x, s)}{-\omega(s)}\int_x^a y_2(t, s) f(t)\, dt,$$

where $G(x,t;s)$ is the Green' s function of the problem under consideration. Using smooth-

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* For simplicity of notation we assume that the eigenvalues are simple. Below we shall show that problem (1) can have only a finite number of multiple eigenvalues.

ness of $f(x)$ and the boundary conditions, one can prove the identity

$$-s\Phi(x,s) - i\,\frac{y_1(x,s)}{\omega(s)}\,f(a) = \frac{f(x)}{s} + \frac{y_1(x,s)}{s\omega(s)}\,f'(a) - \frac{1}{s}G_{sLf}, \tag{9}$$

where G_s is the integral operator with kernel $G(x,t;s)$.

As the contour of integration we choose the rectangle γ_N with vertices $(\pm R_N, T_N)$, $(\pm R_N, -T_N)$, where

$$R_N = \frac{\pi}{a}\left(N + \frac{1}{2}\right) - \frac{1}{2a}\arg A,$$

and T_N is such that $T_N/\ln N \to \infty$, $T_N/R_N \to 0$ as $N \to \infty$.

Using the asymptotic formulas (5)–(7), one can show that, under the conditions of the theorem, the integral over γ_N of the second and third terms on the right-hand side of (9) is $O(N^{-\delta/b})$ on $[0, a-\delta]$. Therefore, by virtue of (9),

$$\frac{1}{2\pi i}\oint_{\gamma_N}\left[-s\Phi(x,s) - i\,\frac{y_1(x,s)}{\omega(s)}\,f(a)\right]ds \to f(x).$$

On the other hand, taking this integral by residues and using the relation

$$\omega'(s_n) = \frac{2s_n}{\varphi_n(a)}\left(\int_0^a \varphi_n^2(t)\,dt - i\,\frac{\varphi_n^2(a)}{2s_n}\right),$$

we obtain the partial sum of the series (4). The theorem is proved.

3°. By the same method, but somewhat more complicatedly, one investigates the case of an arbitrary nonnegative integer μ in formula (2). In this case stronger restrictions have to be imposed on the function being expanded. Denote by D_μ ($\mu \geq 0$ an integer) the class of functions $f(x)$ satisfying the following conditions:

a) the derivative $f^{(\mu+2-k)}(x)$ exists and has bounded variation on the interval $[x_k, a]$, where $x_k = \max[0, a-kb]$, $b = 2a/\mu + 2$, $1 \leq k \leq 2 + [\mu/2]$*;

b) $L^k f(0) = 0$, $0 \leq k \leq [\mu/4]$, $f^{(k)}(a) = 0$, $1 \leq k \leq [\mu]$.

Then the following is valid.

Theorem 2. For $f \in D_\mu$, the series (4) converges to $f(x)$ uniformly on $[0, a-\delta]$, and

$$\left| f(x) - \sum_{n=-N_1}^{N_2} c_n \varphi_n(x) \right| \leq \text{const} \cdot N^{-\delta/b}, \qquad N = \min(N_1, N_2). \qquad (10)$$

In connection with this theorem we make several remarks.

1) By somewhat strengthening the smoothness requirements on $f(x)$, one can obtain uniform convergence of the series on the whole interval $[0, a]$.

2) The smoothness conditions imposed in the theorem on the function $f(x)$ are essential. One can construct a function $f(x)$ satisfying all smoothness requirements except one (for example, $f^{(\mu)}(x)$ has a discontinuity at a point $a - b < x_0 < a$) and for which the series (4) diverges on the whole interval $[0, a]$ or on some part of it.

3) It follows from Theorem 2 that when $\mu = 0$, in the condition of Theorem 1 it is sufficient to require one-time differentiability of $f(x)$.

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* Let us explain that the smoothness requirements on $f(x)$ imposed by this condition weaken as one moves away from the right endpoint of the interval $[0, a]$; thus, on the interval $[a-b, a]$, $f(x)$ has $\mu+1$ derivatives, on the interval $[a-2b, a-b]$ it has μ derivatives, and so on.

4°. Let us note that the system of functions (3) is linearly dependent on $[0, a)$. Namely, as can be shown, for any function $f \in D_{\mu+2m-1}$, $m = 0, 1, \ldots$, the relations

$$\sum_{n=-\infty}^{\infty} s_n^{2j-1} c_n \varphi_n(x) \equiv 0, \qquad x \in [0, a), \qquad 0 \leqslant j \leqslant m, \qquad (11)$$

hold, where c_n are determined from formula (4). (In addition, for $j = 0$ it is necessary to require that $f(a) = 0$.)

Equality (11) makes it possible to establish the following fact:

Theorem 3. *The system of functions (3) is twice complete in $L_2(0, a)$, and for any pair of functions $\{f_0(x), f_1(x)\}$, $f_0 \in D_{\mu+1}$, $f_1 \in D_\mu$, $f_1(a) = 0$, the expansion*

$$f_i(x) = \sum_{n=-\infty}^{\infty} a_n s_n^i \varphi_n(x), \qquad i = 0, 1, \qquad (12)$$

(see (13)), converging uniformly inside the interval $[0, a)$*, is valid.*

Indeed, by Theorem 1,

$$f_0(x) = \sum_{n=-\infty}^{\infty} c_n^{(0)} \varphi_n(x), \qquad f_1(x) = \sum_{n=-\infty}^{\infty} c_n^{(1)} \varphi_n(x),$$

and according to (11)

$$\sum_{n=-\infty}^{\infty} c_n^{(0)} s_n \varphi_n(x) \equiv 0, \qquad \sum_{n=-\infty}^{\infty} c_n^{(1)} \frac{\varphi_n(x)}{s_n} \equiv 0.$$

Therefore

$$f_0(x) = \sum_{n=-\infty}^{\infty} \left(c_n^{(0)} + \frac{c_n^{(1)}}{s_n} \right) \varphi_n(x), \qquad f_1(x) = \sum_{n=-\infty}^{\infty} \left(c_n^{(0)} + \frac{c_n^{(1)}}{s_n} \right) s_n \varphi_n(x),$$

and the theorem is proved for

$$a_n = c_n^{(0)} + c_n^{(1)}/s_n. \tag{13}$$

Unfortunately, in the general case we have not been able to prove the uniqueness of the expansion (12); however, if the expansion (12) converges uniformly on the closed interval $[0, a]$, then it is unique.

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Note: Figure translations are in progress. See original paper for figures.