NECESSARY AND SUFFICIENT CONDITIONS FOR THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM

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Abstract

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM

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The main result of the paper is the proof that, for parabolic equations with constant coefficients whose fundamental solution level surfaces (with respect to $x_1, x_2, \ldots, x_n$, with the time coordinate t fixed) are well arranged and admit a simple description, a necessary and sufficient condition for the pointwise stabilization of the solution of the Cauchy problem is the existence, for the initial function, of a limiting mean over bodies bounded by the level surfaces of the fundamental solution. This assertion is established with the aid of the well-known Tauberian theorem of N. Wiener ([1]). In addition, equations with coefficients depending on t are indicated for which analogous assertions are valid. In conclusion, for general parabolic equations, some properties of the means of the initial function are studied under the assumption that the solution of the Cauchy problem constructed from it stabilizes.

1. As a model, let us consider the equation

$$\partial u/\partial t = (-1)^{b-1}\Delta^b u, \qquad \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2. \tag{1}$$

Its fundamental solution (for $t > 0$) is determined by the formula

$$G(t,x) = (2\pi)^{-n} \int e^{ix\cdot\sigma - |\sigma|^{2b} t}\, d\sigma, \qquad |\sigma|^2 = \sigma_1^2 + \cdots + \sigma_n^2. \tag{2}$$

It is not difficult to verify that $G(t,x)$ in fact depends on two variables t and r and is determined by the formula

$$G(t,x) \equiv G_1(t,r) =$$

$$= t^{-n/2b} k_n \int_0^\infty \rho^{n-1} e^{-\rho^{2b}} J_{n/2-1}\Big(\rho \frac{r}{t^{1/2b}}\Big) \Big(\rho \frac{r}{t^{1/2b}}\Big)^{1-n/2} d\rho = t^{-n/2b} G_2\Big(\frac{r}{t^{1/2b}}\Big); \tag{3}$$

$J_\nu(x)$ is the Bessel function of the first kind of order ν ([3]). We are interested in the question of when there exists, as $t \to \infty$, a limit of the solution of the Cauchy problem

$$u\big|_{t=0} = u_0(x) \tag{4}$$

for equation (1), where $u_0(x)$ is a continuous bounded function, determined by the Poisson integral

$$u(t,x) = \int G(t, x-\xi) u_0(\xi)\, d\xi. \tag{5}$$

Theorem 1. *In order that* $\lim\limits_{t\to\infty} u(t,x) = l$, *it is necessary and sufficient that the initial function have a spherical limiting mean equal to* l, *i.e.*

$$\lim_{R\to\infty} \frac{1}{\operatorname{mes} K_R} \int_{K_R} u_0(x)\, dx = l; \tag{6}$$

K_R is the ball of radius R with center at the point x.

Proof. One may always assume $l = 0$. Let us write integral (5), $x = 0$, in spherical coordinates:

$$u(t,0) = t^{-1/2b} \int_0^\infty \left(\frac{r}{t^{1/2b}}\right)^{n-1} G_2\left(\frac{r}{t^{1/2b}}\right) v_0(r)\, dr;$$

$v_0(r)$ is the integral of $u_0(\xi)$ over the sphere of radius r, $n \geq 2$.

Consider

$$I_2 v_0 = t^{-1/2b} \int_1^\infty \left(\frac{r}{t^{1/2b}}\right)^{n-1} G_2\left(\frac{r}{t^{1/2b}}\right) v_0(r)\, dr.$$

It is obvious that $u(t,0)$ and $I_1 v_0$ tend to zero as $t \to \infty$ simultaneously. Along with $I_2 v_0$ consider

$$I_1 v_0 = \frac{1}{t^{1/2b}} \int_0^{t^{1/2b}} v_0(r)\, dr$$

and in both integrals introduce new variables $r = e^y$, $e^{1/2b} = e^\eta$. Then

$$I_{sv} 0 = \int_{-\infty}^\infty K_s(\eta - y) v_0(y)\, dy,$$

where

$$K_1(\eta) = \begin{cases} e^{-\eta}, & \eta \geq 0, \\ 0, & \eta < 0, \end{cases} \qquad K_2(\eta) = e^{-n\eta} G_2(e^{-\eta}), \qquad v_0(y) \equiv 0, \quad y < 0.$$

It is obvious that the kernels $K_s(\eta) \in L_1(-\infty, \infty)$ and that the Fourier transform of the kernel $K_1(\eta)$ has no real zeros; in order to establish the analogous fact for the kernel $K_2(\eta)$, write its Fourier transform in the form

$$F(z) = k_n \int_0^\infty r^z \left(\int_0^\infty \alpha^{n/2} J_{n/2-1}(\alpha r) e^{-\alpha^{2b}} \, d\alpha \right) dr, \qquad z = \frac{n}{2} + ix, \quad -\infty < x < \infty.$$

It turns out that, with the aid of formula (1) from ((3), p. 428), the function $F(z)$ is explicitly computed for $-1 < \operatorname{Re} z < 0$, and then analytically continued to the required values of the argument. In the case $n = 1$, $F(K_2(\eta))$ is also computed. It then turns out that, for any n and any real values of the argument, $F(K_2(\eta)) \neq 0$. Therefore, by N. Wiener' s Tauberian theorem (1),

$$\lim_{t \to \infty} u(t, 0) = 0$$

if and only if

$$\lim_{R \to \infty} R^{-1} \int_0^R v_0(r) \, dr = 0.$$

From the last assertion theorem 1 is established with the aid of the following elementary lemma.

Lemma 1. The limits

$$\lim_{R \to \infty} n R^{-n} \int_0^R r^{n-1} v_0(r) \, dr, \qquad \lim_{R \to \infty} R^{-1} \int_0^R v_0(r) \, dr,$$

where $v_0(r)$ is a bounded continuous function, exist simultaneously and their values coincide.

2. We give a generalization of theorem 1. Its proof, in its essential part, uses theorem 1.

Theorem 2. Let the coefficients of the parabolic equation

$$\frac{\partial u}{\partial t} = (-1)^{b-1} \left(\sum_{i,j=1}^n a_{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} \right)^b u \tag{7}$$

satisfy the conditions:

$$1) \quad \lim_{t \to \infty} \bar{A}_{ij}(t) [\det A(t)]^{1/n} = \alpha_{ij}, \quad \alpha_{ii} \neq 0;$$

$$2) \quad \lim_{t\to\infty} \det A(t) = \infty;$$

$$A(t) = \left\| \int_0^t a_{ij}(\tau)\,d\tau \right\|, \qquad \bar{A}_{ij}(t) \text{ are the elements of the matrix inverse to } A(t).$$

Then, in order that the solution of problem (7), (4), represented by the Poisson integral, tend to the limit l, it is necessary and sufficient that

$$\lim_{r\to\infty} \frac{1}{\operatorname{mes} F_r} \int_{(F_r)} u_0(\xi)\,d\xi = l;$$

(F_r) is the body bounded by the surface

$$\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j = r^2.$$

An analogous assertion is valid for the equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t)\frac{\partial u}{\partial x_i} + c(t)u, \tag{8}$$

if, in addition to assumptions 1) and 2) of Theorem 2 concerning $a_{ij}(t)$, the following conditions are satisfied:

$$3) \quad \lim_{t\to\infty} B_i(t)[\det A(t)]^{-1/n} = 0;$$

$$4) \quad \lim_{t\to\infty} C(t) = \gamma;$$

$$B_i(t) = \int_0^t b_i(\tau)\,d\tau, \qquad C(t) = \int_0^t c(\tau)\,d\tau,$$

γ being any number. The stabilization will be to the limit le^{γ}.

Thus, in the case of pointwise stabilization of the solution of the Cauchy problem it is necessary that the initial function have a limiting mean over a quite definite system of bodies. How delicately the limiting means take into account the behavior of the initial function at infinity may be judged from the following.

Theorem 3. 1) Let F_R and Φ_R be families of bodies obtained by a similarity transformation with center at the origin and similarity coefficient R of two different bodies containing the origin. Then one can specify a bounded continuous function $u_0(x)$ having a limiting mean over one family and not having a mean over the other.

2) Let F_R and Φ_R be two properly arranged families of bodies*, for which

$$\lim_{R\to\infty} \frac{\operatorname{mes}(F_R \cup \Phi_R / F_R \cap \Phi_R)}{\operatorname{mes}(F_R \cap \Phi_R)} = 0.$$

Then for any bounded locally summable function the limiting means over the systems F_R and Φ_R exist simultaneously.

Let us also note the following proposition.

Theorem 4. If $u_0(x)$ has a limiting mean equal to l over some properly arranged system of bodies and, for $|x| > R$, $u_0(x) \geq l$ (or $u_0(x) \leq l$), then $u_0(x)$ has a limiting mean over any properly arranged system of bodies. Therefore the solution of the Cauchy problem for equations (7), (8), constructed from this initial function, stabilizes as $t \to \infty$.

In conclusion we give a one-dimensional version of the stabilization theorem for arbitrary parabolic equations of higher orders whose Green functions possess properly arranged families of level surfaces.

* That is, families of convex bodies F_R bounded by surfaces $\bar{F}_R$, satisfying the following conditions: 1) exactly one surface passes through each point of space; 2) there exists a constant k such that, for any surface of this family, the ratio of the greatest radius vector to the least does not exceed k.

Consider the function $u(t,x) = \int G(t, x-\xi) u_0(\xi)\, d\xi$ and suppose that $G(t,x)$ satisfies the following conditions:

1) $\displaystyle\int G(t,x)\, dx = 1;$

2) $G(t,x) = G(t,-x);$

3) $|D^{mG}(t,x)| \leq \dfrac{C}{a(t)} \left(1 + \dfrac{|x|}{a(t)}\right)^r, \quad m = 0,1,\ r < -1, \quad a(t) \to \infty, \quad t \to \infty.$

Theorem 5. I. If the double limit

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{t\to\infty}}\ m \left| \frac{1}{2t} \int_{-t}^{t} u_0(x^m)\, dx \right| = 0,$$

then

$$\lim_{t\to\infty} u(t,x) = 0.$$

II. If

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{t\to\infty}}\ m\left|\frac{1}{2t}\int_{-t}^{t}u_0(x^m)\,dx\right|$$

is greater than a constant δ, depending on the coefficients of the equation, then

$$\lim_{t\to\infty}u(t,x)\neq 0.$$

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Note: Figure translations are in progress. See original paper for figures.