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Abstract

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ON A PROBLEM OF OPTIMAL CONTROL OF STATIONARY SYSTEMS WITH DISTRIBUTED PARAMETERS

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In this note some new theorems are formulated in the theory of optimal processes described by n-dimensional linear differential equations of parabolic type. As is known, the case of systems with distributed parameters, owing to the difficulties inherent in it, has been studied comparatively little. Some such problems (for the simplest equation $u'_t = u''_{xx}$) have been studied, for example, in (^{1,2}), (see also (^{3,4})).

1. Statement of the problem. In the cylinder $Q(T) = \Omega \times (0, T)$, where $\Omega \in R^n$ is an n-dimensional bounded domain with piecewise smooth boundary Σ, a parabolic equation is given

$$L(u) \equiv b(p)u'_t(t,p) - \sum_{i,j} \frac{\partial}{\partial x_j} \left(A_{ij}(p) \frac{\partial u}{\partial x_i} \right) - c(p)u(t,p) = 0 \qquad (1)$$

with measurable bounded (in Ω) time-independent coefficients $b(p)$, $A_{ij}(p)$, $c(p)$, where: a) $\operatorname{vrai\,inf}_{p\in\Omega} b(p) \geq b_0 > 0$, $p = \{x_1, \dots, x_n\}$, b) $\sum_{ij} A_{ij}(p)\alpha_i\alpha_j \geq \nu \sum_j \alpha_j^2$. It may also be assumed (without loss of generality) that $c(p) \leq -c_0 < 0$. Let M be an arbitrary bounded closed set of some vector space R^s*. Measurable vector-functions of time $\vec{\mu}(t)$, $t_0 < t \leq t_1$, with values in the set M, will be called admissible controls, forming the functional class D (^{4}). Accordingly, the set M will be called the control domain. Further, let $g(p, \vec{\mu})$, $p \in \Sigma$, $\vec{\mu} \in M$, be a certain bounded function, continuous in $\vec{\mu}$ (for fixed p) and measurable in p (for fixed $\vec{\mu}$); $a(p)$ a nonnegative measurable function**, defined on Σ. For a given control $\vec{\mu}(t) \in D$ and zero initial conditions $u(t,p)\big|_{t=0} = 0$, there exists (see (^{5})) a unique generalized solution $u(t,p)$ of the third boundary-value problem for equation (1), satisfying, for any $\Phi(t,p) \in W_2^1$, the integral identity

$$\int_\Omega b(p)u_\mu(t,p)\Phi(t,p)\,d\Omega - \int_{Q(t)} b(p)u_\mu \Phi'_t\,dQ +$$

$$+\int_{Q(t)}\sum_{i,j}A_{ij}u_{\mu x_i}\Phi_{x_j}\,dQ-\int_{Q(t)}cu_\mu\Phi\,dQ+\int_{S(t)}[au_\mu-g(p,\vec{\mu}(t))]\Phi\,dS=0,$$

where $S(t)=\Sigma\times(0,T)$, $0\leq t\leq T$.

* The subsequent arguments (up to and including paragraph 3) go through without substantial changes also for sets M that are subsets of more general spaces.

** The function $a(p)$ is also assumed bounded.

It can be proved (see (5)) that the function $u_\mu(t,p)$ has Hölder continuity in the cylinder $Q(T)=\Omega\times(0,T]$, and the a priori estimate

$$\sup_{\left\{\begin{array}{c}\vec{\mu}(t)\in D\\(t,p)\in Q(T)\end{array}\right\}}|u_\mu(t,p)|\leqslant c$$

holds.

Taking as the control criterion the functional

$$I_1(M)=\int_\Omega F_1(p,u_\mu)\,d\Omega$$

or

$$I_2(\mu)=\int_\Omega F_2(p,u_\mu,\nabla u_\mu)\,d\Omega\,^*,$$

where $F_1\geqslant 0$, $F_2\geqslant 0$ are functions measurable in p and continuous in u (in addition, for F_2 the convexity condition with respect to the components of the vector ∇u_μ is fulfilled), we consider the following optimal problem: among all controls $\vec{\mu}(t)\in D$, find such a $\vec{\mu}_0(t)\in D$ for which the equality

$$I_1(\mu_0)=\inf_{\vec{\mu}\in D}I_1(\mu)$$

or

$$I_2(\mu_0)=\inf_{\vec{\mu}\in D}I_2(\mu),$$

holds, where $0\leqslant t\leqslant T$; T is fixed.

2. For the case of linear controls the following holds.

Existence theorem. If

$$g(p, \vec{\mu}) = \sum_{j=1}^{s} g_j(p)\mu_j,$$

where $g_j(p)$, $p \in \Sigma$, are bounded measurable functions; if M, the range of control, is a convex set and there exists at least one control $\vec{\mu}(t) \in D_1$ for which $I_1(\mu) < +\infty$ (or $I_2(\mu) < D$), then there also exists an optimal control $\vec{\mu}_0(t) \in D$, realizing

$$\inf_{\mu \in D} I_1(\mu)$$

(or

$$\inf_{\mu \in D} I_2(\mu)$$

).

3. **Necessary optimality criterion.** Let $\{v_m(p)\}$ be a complete orthonormal system in $L_2(\Omega)$ of generalized eigenfunctions satisfying, for any $\psi(p) \in W_2^1(\Omega)$, the integral identity

$$\int_{\Omega} \left\{ \sum_{ij} A_{ij}(p) v_{mx_i} \psi_{x_j} - c(p) v_m(p) \psi(p) \right\} d\Omega + \int_{\Sigma} a v_m \psi \, dS = \lambda_m \int_{\Omega} b(p) v_m \psi \, d\Omega.$$

The following necessary optimality criterion holds for controls $\vec{\mu}_0(t) \in D$ realizing $\inf_{\mu \in D} I_1(\mu)$ (in the linear case of our problem, both with fixed and with nonfixed time): if $\mu_0(t)$ is optimal, then

$$\inf_{\mu \in M} \sum_{j=1}^{s} \Phi_j(\tau)\mu_j = \sum_{j=1}^{s} \Phi_j(\tau)\mu_{0j}(\tau) \tag{2}$$

(for almost all $\tau \in (0, T')$), where

$$\Phi_j(\tau) = \sum_m F_m(\vec{\mu}_0) a_{mj} e^{-\lambda_m (T-\tau)},$$

with

$$F_m(\vec{\mu}_0) = \int_{\Omega} F'_{1u}(p, u_{\mu_0}(T, p)) v_m(p) \, d\Omega, \qquad a_{mj} = \int_{\Sigma} g_j(p) v_m(p) \, dS.$$

It is assumed that

$$\int_{\Omega} \{F'_{1u}\}^2 d\Omega < C \quad \text{for} \quad |u| < \bar{C},$$

where

$$\overline{C} \geq \sup_{\left\{ \substack{\vec{\mu}\in D \\ p\in\Omega} \right\}} |u_{\mu}(T,p)|.$$

On the basis of criterion (2), theorems can be proved that characterize the properties of the optimal control $\mu_0(t) \in D$: a) a switching theorem, b) uniqueness theorems for the optimal control.

4. Switching theorem. Suppose that the control region M is an s-dimensional convex bounded polyhedron $\Pi^s \subset R^s$ (so that $\mu_0(t) \in \Pi^s$ for almost all $t \in (0,T)$). Then, if for all α the function

$$F^{(\alpha)}(\tau) \equiv \sum_{m=1}^{\infty} F_m(\vec{\mu}_0) \left\{ \sum_{j=1}^{s} a_{mj} l_j^{(\alpha)} \right\} e^{-\lambda_m (T-\tau)} \neq 0 \tag{3}$$

(here $\{l_j^{(\alpha)}\}$ are the direction cosines of the noncollinear edges $\vec{l}^{(\alpha)}$ of the polyhedron Π^s), then $\vec{\mu}_0(t)$ is a piecewise-constant vector-function (on each interval of the form $(0, T-\delta)$, $\delta > 0$) with values at the vertices of the polyhedron Π^s (i.e., there are no more than finitely many switching points from vertex to vertex on each interval of the form $(0, T-\delta)$).

Let us give one sufficient criterion for condition (3) to hold. Let, for example: 1) $\sum_j a_{mj} l_j^{(\alpha)} \neq 0$ for all $m = 1, \dots, \infty$ and all α; 2)

$$\inf_{\vec{\mu}\in D} \int_{\Omega} \{F'_{1u}(p, u_{\mu}(T,p))\}^2 \, d\Omega > 0;$$

3) all λ_m are distinct. Then (3) will be satisfied, since, by Parseval' s equality,

$$\sum_{m=1}^{\infty} F_m^2(\mu_0) = \int_{\Omega} \{F'_{1u}(p, u_{\mu_0}(T,p))\}^2 \, d\Omega > 0,$$

and, consequently, for some $m = m_0$, $F_{m_0}(\mu_0) \neq 0$ (if $F^{(\alpha)}(\tau) \equiv 0$, then from conditions 1), 3) of this item it would follow that all $F_m(\mu_0) = 0$). It can be

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* In the case of the functional I_2, it is necessary to assume additionally that the generalized derivatives $b_{x_j}(p)$ and $A_{ijx_k}(p)$ are bounded in Ω. This entails the Hölder continuity of $u_{x_j}(t,p)$ (see (5)).

shown that for "almost all" sets $\{g_j(p)\}$, condition 1) of this item will hold; analogous statements are also valid with respect to conditions 2) and 3).

Let us consider an example. Suppose that, for the equation $u'_t = u''_{xx}$ with zero initial conditions, the third boundary-value problem is solved, where

$$u'_x(t,x)\big|_{x=0} = 0, \qquad u'_x(t,x)\big|_{x=1} = a(\mu(t) - u(t,x))\big|_{x=1}, \tag{4}$$

where $a > 0$, $|\mu(t)| \leq 1$, $0 \leq t \leq T$. Let, further,

$$I(\mu) = \int_0^1 [u_\mu(T,x) - u_0(x)]^2\,dx \qquad (u_0(x) \in L_2(0,1)).$$

It is not difficult to see that if $\sup_{|\mu|\leq 1} I(\mu) > 0$, then there exists $m = m_0$ such that

$$F_{m_0}(\mu_0) = 2\int_0^1 (u_{\mu_0}(T,x) - u_0(x))\,v_{m_0}(x)\,dx \neq 0,$$

where $\{v_m(x) = \cos\sqrt{\lambda_m}x\}$ is the system of eigenfunctions of the operator u''_{xx} under the boundary conditions (4), and all λ_m are distinct. Since in the present case

$$F^{(1)}(\tau) = \sum_m F_m(\mu_0) a_m e^{-\lambda_m(T-\tau)},$$

where $a_m = a\|v_m\|_{L_2(0,1)} \times v_m(1) \neq 0$, it follows that $F^{(1)}(\tau) \neq 0$, and one may assert that the optimal control $\mu_0(t)$ is piecewise constant on each interval of the form $(0, T-\delta)$ and satisfies $|\mu_0(t)| = 1$.

5. The case of unfixed time. In the case where the time T is not fixed, to the basic relation (2) one can add a relation containing the unknown time T. Indeed, by means of variations of a two-sided shift one can prove the following identity for the optimal control $\vec{\mu}_0(t)$:

$$\sum_m F_m(\vec{\mu}_0)\lambda_m \left\{\int_0^\tau \sum_j \mu_{0j}(t) a_{mj} e^{-\lambda_m(T-t)}\,dt\right\} \equiv \sum_m F_m(\vec{\mu}_0) \sum_j \mu_{0j}(\tau) a_{mj} e^{-\lambda_m(T-\tau)} \tag{5}$$

for almost all $\tau \in (0,T)$, or, using the notation

$$\Phi_j(\tau) \equiv \sum_m F_m a_{mj} e^{-\lambda_m(T-\tau)}$$

(see (2)), (5) can be rewritten in the form

$$\sum_j \int_0^\tau \Phi'_{jt}(t)\mu_{0j}(t)\,dt \equiv \sum_j \Phi_j(\tau)\mu_{0j}(\tau) \tag{6}$$

for almost all $\tau \in (0, T)$. In particular, for example, if the control $\vec{\mu}_0(t)$ is piecewise constant on $(0, T-\delta)$, then, putting in (6) $\tau = 0$, we shall have

$$\sum_j \sum_m F_m(\vec{\mu}_0) a_{mj} c_{je}^{-\lambda_{mT}} = 0,$$

where $\vec{c} = \{c_j\}$ is one of the vertices of the polyhedron Π^s.

6. Uniqueness theorems.

Theorem 1. Let the function $F(p, u)$ be convex in u (for any fixed $p \in \Omega$), i.e., possess the property

$$F\left(p, \sum_j \theta_j u_j\right) \leq \sum_j \theta_j F(p, u_j)$$

(for any u_j with $\theta_j \geq 0$, $\sum_j \theta_j = 1$), and let conditions 1), 2), 3) of the switching theorem be satisfied. Then any two optimal controls $\mu_{0j}(t)$, $j = 1, 2$ (in the problem with fixed time), coincide almost everywhere in $(0, T)$.

Theorem 2. Let the function $F(p, u)$ possess the property: from the equalities

$$\int_\Omega F(p, u_{\mu_1})\,d\Omega = \int_\Omega F(p, u_{\mu_2})\,d\Omega = \inf_{\vec{\mu} \in D} \int_\Omega F(p, u_\mu)\,d\Omega,$$

where $\vec{\mu}_1 \in D$, $\vec{\mu}_2 \in D$, it follows that

$$u_{\mu_1}(T, p) \equiv u_{\mu_2}(T, p)$$

(for almost all $p \in \Omega$). Then, under fulfillment of the same conditions 1), 2), 3) of the switching theorem, the optimal control of our problem with fixed time will be determined uniquely (up to values on a set of measure zero).

The main role in the proof of Theorem 2 is played by

Lemma. Whatever the function $g(p) \in L_2(\Sigma)$, there exists a function $\psi(p) \in W_2^1(\Omega)$ such that

$$\sum_{m=1}^{\infty} \left(\int_\Sigma g v_m\,dS\right)^2 \Big/ \lambda_m^2 = \int_\Omega b(p)\psi^2(p)\,d\Omega.$$

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Note: Figure translations are in progress. See original paper for figures.