ON THE DIMENSION OF THE REMAINDERS OF PROXIMITY AND TOPOLOGICAL SPACES

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MATHEMATICS

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ON THE DIMENSION OF THE REMAINDERS OF PROXIMITY AND TOPOLOGICAL SPACES

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As early as 1954, in the paper (1) (in detail in (2)), a necessary and sufficient condition was given for the **proximity dimension** $\delta \dim N_c$ of the remainder $N_c = cP \setminus P$ of a proximity space* P in its natural bicompact extension cP to be no greater than a given number n. This condition was expressed, of course, in proximity terms of the given space P itself and consisted in the following:

Theorem 0. $\delta \dim N_c \leq n$ if and only if into every proximity framing of the space P one can inscribe a proximity framing of multiplicity $\leq n+1$.

Here a **proximity framing** (1, 2) means any finite system of open sets $\Gamma_1, \dots, \Gamma_k$ such that the complement
$P \setminus \Gamma_1 \setminus \dots \setminus \Gamma_k$ is bicompact, and which satisfies a certain rather special additional condition, which we shall omit. In 1964 Isbell (3) translated Theorem 0 into the language of the theory of uniform (Weil) structures (see Theorem 00).

It was natural to pose the question of finding such a condition that would give not the proximity dimension $\delta \dim N_c$ of the remainder N_c, but the ordinary topological dimension $\dim N_c$, defined by means of coverings. As it turned out (see (4)), this question is resolved in a similar way—one need only use not proximity framings, but so-called framings.

Definition 1. A **prolongable framing** of a proximity space P is any system of open sets $\Gamma_1, \dots, \Gamma_k$ such that the complement
$P \setminus \Gamma_1 \setminus \dots \setminus \Gamma_k$ is bicompact and, moreover, for every neighborhood H of this complement the system $\{\Gamma_1, \dots, \Gamma_k, H\}$ is a proximity covering (see (8), p. 559) of the space P.

It turns out that the following new result holds.

Theorem 1. $\dim N_c \leq n$ if and only if into every prolongable framing of the space P one can inscribe a prolongable framing of order $\leq n+1$.

We note that if one inscribes prolongable framings of multiplicity $\leq n+1$ not into prolongable framings, but into proximity coverings, then one obtains another dimensional characteristic of the remainder N_c, a characteristic that somewhat generalizes the so-called **metric dimension** ([6], [7]) in the sense of P. S. Aleksandrov to arbitrary proximity spaces.

Definition μ. We shall say that the **metric characteristic** $\mu \dim Q$ of a proximity space Q does not exceed n if into each of its proximity coverings one can inscribe a covering of multiplicity** $\leq n+1$.

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* By a proximity space, or space of proximity, we mean here proximity spaces in the sense of V. A. Efremovich ([5]).

** We consider only finite open coverings of normal spaces and the so-called normal finite open coverings of completely regular non-normal spaces (see ([1]), p. 719, or the last paragraph in ([2])).

Theorem 2. $\mu \dim N_c \leq n$ if and only if into every proximity covering of the space P one can inscribe a continued bordering of multiplicity $\leq n+1$.

Theorem 0 in fact gives us a necessary and sufficient condition for the dimension $\dim \overline{N}_c$ of the closure of the growth N_c in the extension cP to be no greater than n. This, it seems to me, is more interesting than what Theorem 2 gives us. Still more interesting and important to me is Theorem 1, not only because it "gives" us the dimension $\dim N_c$ of the growth N_c itself, but also because it leads us to the solution of the following topological problem:

When does a given topological space X possess a bicompact extension cX with growth $N_c = cX \setminus X$ of dimension $\leq n$?

Of course, only completely regular spaces are meant here.

As is known, this problem, even in the case of spaces with a countable base, had been solved only for $n = 0$ ([9–13]). Freudenthal first solved this problem for spaces with a countable base ([9]), having in mind bicompact extensions that also have a countable base, and then gave a solution for the general case as well ([10]). The proof he gave for the general case turned out not to be quite rigorous. K. Morita ([11]) gave a rigorous proof of the sufficiency of the condition for spaces of arbitrary weight. It was later observed ([14]) that in the general case of completely regular spaces this condition is not necessary. E. Sklyarenko, for spaces of a certain very broad class $\mathfrak{C}$, proved ([12,13]) that the condition of semibicompactness (otherwise, peripheral bicompactness) found by Freudenthal is a necessary and sufficient condition for the space X to have a bicompact extension with inductively zero-dimensional growth.

Let us note the following two circumstances.

First, Freudenthal (and after him Morita, Sklyarenko, de Groot ([15]), de Vries ([16]), and some others) are concerned with the inductive dimension of growths, not with the dimension dim, and these, as is known, are in general different things ([17–19]). Second (apparently by force of this circumstance), the above-mentioned authors seek an inductively defined condition on the space X (see, for example, de Groot' s seminar ([15]) or de Vries' s dissertation ([16]))* . Only by a certain, one might say, fortunate accident for the Freudenthal–Sklyarenko theorem did it turn out that for spaces of the class $\mathfrak{C}$ all growths N_c are finally compact ([12,13]) and, consequently, the equalities $\operatorname{ind} N \leq 0$, $\operatorname{Ind} N \leq 0$, and $\dim N \leq 0$ are equivalent ([20]).

We, I repeat, are interested in the dimension of growths as defined by means of coverings, and therefore we need other conditions as well.

These other conditions are the following. First of all, we need the concept of a bordering.

Definition 2. By a **bordering** of a space X we mean any continued bordering of this space X, considered in its maximal (Čech) proximity. For normal spaces the notion of a bordering is especially simple: a **bordering of a normal space** X is simply any such system of open sets $\Gamma_1, \dots, \Gamma_k$ that the complement $X \setminus \Gamma_1 \setminus \dots \setminus \Gamma_k$ is bicompact.

Further, we have to abandon the condition of inscribing borderings, since otherwise, according to Theorem 1, we obtain only a characterization of the dimension of the Čech growth $N_\beta = \beta X \setminus X$.

Corollary (of Theorem 1). $\dim N_\beta \leq n$ if and only if into every bordering of the space X one can inscribe a bordering of multiplicity $\leq n+1$.

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* Sklyarenko alone sought conditions of another kind. However, his condition, apparently, is only necessary ([21]).

This may be good, but it is not what we need now. Let us give two more important definitions.

Definition 3. We shall call a **structure of rims** any family $\Sigma = \{\gamma\}$ such that, for any two rims γ' and γ'' of this family, there is a rim γ of the same family Σ such that γ is star-refined* into both γ' and γ''.

Definition 4. We shall say that a **family** $\Sigma = \{\gamma\}$ **has the basis property** if, for every point x of the given space X and for every neighborhood Ox of it, there are a rim γ in the family Σ and a neighborhood Ux of the point x such that the γ-star of the neighborhood Ux lies entirely in Ox: $\operatorname{St}_\gamma Ux \subseteq Ox$.

We now state our first main theorem.

Theorem 3. *A space X has a bicompact extension cX with remainder of dimension $\leq n$ if and only if in X there exists at least one structure of rims of multiplicity $\leq n+1$ having the basis property.*

This theorem is new even for spaces with a countable base and even for $n = 0$, since our condition, although equivalent to the condition of semibicompactness, nevertheless has an entirely different character. Thus, the desired generalization of the Freudenthal–Sklyarenko theorem has been obtained.

It is now time to pose the question of generalizing the original Freudenthal theorem to the case of an arbitrary number n. It turns out that this too is possible, and not only for countable, but also for arbitrary weight.

Theorem $3'$**.** *A space X of weight τ has a bicompact extension cX of weight τ and with remainder of dimension $\leq n$ if and only if in X there exists at least one structure Σ of rims of multiplicity $\leq n+1$, having the basis property.*

Let us note that, in the condition of this theorem, one may require that the structure Σ have cardinality $\leq \tau$.

For spaces of countable weight the condition can be somewhat simplified.

Theorem $3''$**.** *A space X of countable weight has a bicompact extension of countable weight with remainder of dimension $\leq n$ if and only if in X there exists at least one such countable sequence of rims γ_i of multiplicity $\leq n+1$, having the basis property, that each rim γ_{i+1} of this sequence is star-refined into the preceding rim γ_i.*

Theorem $3'$ follows from Theorem 3 and one more theorem.

Theorem 4. *If a space X of weight τ has some bicompact extension with remainder of dimension $\leq n$, then it also has a bicompact extension cX of the same weight τ and with remainder N_c of dimension $\leq n$.*

Finally, just as in the zero-dimensional case, there is a generalization of the well-known result of Morita–Sklyarenko on the existence of a maximal extension c_0X with zero-dimensional remainder:

Theorem 5. *If a space X has a bicompact extension cX with remainder of dimension $\leq n$, then among all such extensions there exists a greatest one.*

We denote this greatest extension by c_{nX}. Of course, the extensions c_{nX} for different n need not themselves be different (for example, for any discrete space D one always has $c_{nD} = \beta D$ for every n).

The remaining results are auxiliary in character, and we shall omit them. We shall make only one necessary point.

Remark. Apart from Theorem 0, the results presented here in the gen–

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* A system $\gamma = \{\Gamma\}$ is **star-refined** into a system ω if the system of γ-stars $\mathrm{St}_\gamma \Gamma$ of all sets Γ from γ is refined into ω, where the γ-star $\mathrm{St}_\gamma U$ of a set U is the sum (union) of all those elements of the system γ which have nonempty intersection with U.

in the general case of completely regular spaces may fail to hold (in this case Theorems 1, 3, and $3'$ are certainly false). Theorems 3, $3'$, 4, and 5 have been proved by us for all spaces of the above-mentioned class $\mathfrak{C}$. This is a rather extensive class: it includes all locally metrizable spaces and all spaces complete in the sense of Čech [22] (absolute G_δ' s). Theorems 1 and 2 have been proved by us in an even broader class of spaces, namely for all such proximity spaces P that are normally adjoined [4] to their remainder $N_c = cP \setminus P$. Let us explain the words "normally adjoined."

Definition 5. We shall say that a space X is **normally adjoined to its remainder** $N_c = cX \setminus X$ with respect to a given extension cX, if any two closed disjoint sets in the remainder have disjoint open neighborhoods in the whole extension cX. This is, so to speak, the "external" normality of the remainder N_c in cX.

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Note: Figure translations are in progress. See original paper for figures.