A GENERALIZATION OF THE THEOREMS OF CARTER AND WIELANDT

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A GENERALIZATION OF THE THEOREMS OF CARTER AND WIELANDT

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§ 1. The purpose of this note is to generalize two well-known theorems of the theory of finite groups—the theorem of Carter ([1]) and the theorem of Wielandt ([5]). We precede each theorem by several simple auxiliary propositions, which makes it possible to avoid repetitions in the proofs.

We use the following notation and definitions. $N(H)$ and $C(H)$ are, respectively, the normalizer and the centralizer of the complex H in the group G; π and π' are complementary sets of primes. A group is called **π-separable** ([2]) if its π-Hall subgroup is nilpotent and is a direct factor. A nilpotent subgroup that coincides with its normalizer (in G) will be called a **Carter subgroup** of the group G. We shall say that a group G has the **Carter property** if it contains exactly one conjugacy class of Carter subgroups. For example, the simple group of order 168 has the Carter property.

§ 2. **Lemma 1.** *Let H be a Carter subgroup of the group G, and let the subgroup F containing H have the Carter property. Then $N(F) = F$.*

Indeed, applying the Frattini argument gives

$$F \cdot [N(H) \cap N(F)] = N(F).$$

But the left-hand side of this equality, in view of $N(H) = H$, coincides with F.

Lemma 2. *Let the condition of Lemma 1 be satisfied for $F = HM$, where H is a Carter subgroup in G, and M is such a normal divisor of the group G that the factor group G/M is nilpotent. Then $HM = G$.*

This is an immediate consequence of Lemma 1 and the normalizer property of nilpotent groups.

Lemma 3. *If all subgroups of the group G (including G itself) have the Carter property, then the homomorphic images of any of its subgroups have this same property.*

Proof. Obviously, it is enough to show that all homomorphic images of the group G have the Carter property. Let H be a Carter subgroup in G, and let M be a normal divisor in G. By Lemma 1, the subgroup HM/M is a Carter subgroup in G/M. Let L/M be another Carter subgroup in G/M. By assumption, L contains a Carter subgroup F; $L = FM$ by Lemma 2. Further, from the inclusion $N(F) \subseteq N(FM) = L$ it follows that $N(F) = F$. Therefore in G there is an element x such that $H^x = F$. But then $(HM)^x = H^x M = FM = L$, i.e. HM and L are conjugate in G. All the more so, HM/M and L/M are conjugate in G/M, as was required.

Lemma 3a. *Let M be a normal divisor in G; let L/M be a Carter subgroup in G/M; and let H be a Carter subgroup in L. If HM has the Carter property, then $N(H) = H$.*

Indeed, by Lemma 2 we have $HM = L$. From the inclusion $N(H) \subseteq N(HM) = L$ it follows that $N(H) = H$, as required.

For the proof of Theorem 1 the following special case of it is useful:

Lemma 4. *Let R be a nilpotent π-Hall subgroup of the group G, and let the π'-Hall subgroup be invariant in G. If all π'-sub-*

subgroups of the group G have the Carter property, then G also has the Carter property.

Proof. By hypothesis $N(R) = R \times M$. Let F be a Carter subgroup in the π'-subgroup M. Put $H = R \times F$. We have $N(H) \cap N(R) = H$. Since $N(H) \subseteq N(R)$, it follows that $N(H) = H$. Now let H_1 be another Carter subgroup in G. Using Lemma 2, we may, without loss of generality, assume that $R \subseteq H \cap H_1$. Then $N(R) \supseteq \{H, H_1\}$. The subgroups H/R and H_1/R coincide with their normalizers in $N(R)/R$; therefore they are conjugate there. Hence H and H_1 are conjugate in G. Theorem 1 shows that a result analogous to Carter's theorem holds for a broader class of groups than that for which Carter' s theorem was proved.

Theorem 1. *If all π'-subgroups of a π-solvable group G have the Carter property, then the group G itself has this property.*

Proof. We may assume that the theorem has already been proved for all proper subgroups of the group G. By Lemma 3, all nontrivial homomorphic images of the group G then have the Carter property (here induction is also used). We may assume that the set π is nonempty (otherwise the conclusion is weaker than the hypotheses). Then G is not simple. A minimal normal divisor M of the group G is an ω-subgroup, where ω is one of the sets π, π'.

Let A/M be a Carter subgroup in G/M. The subgroup A, by Lemma 4, contains a Carter subgroup H, and moreover A, by the same lemma, has the Carter property. Now by Lemma 3a we have $N(H) = H$.

Now let H_1 and H_2 be distinct Carter subgroups of the group G. By Lemmas 4 and 1, then $N(H_{iM}) = H_{iM},\ i = 1, 2$; hence H_1M/M and H_2M/M are Carter subgroups in G/M, and by induction they are conjugate there. Therefore H_1M and H_2M are conjugate in G. Thus $H_1^x \subseteq H_2M$, x from G. But then H_1^x and H_2 are conjugate in H_2M, by Lemma 4. Hence H_1 and H_2 are conjugate in G, as required.

Example. Let, in a π-solvable group G, its π'-Hall subgroup be isomorphic to $LF(2,7)$. Then G, by Theorem 1, has the Carter property.

Carter' s theorem is, evidently, a special case of Theorem 1. Lemma 5 is, of course, known, but is included for convenience of reading.

Lemma 5. *Let A_0 and B be subgroups of the group G, with B invariant in G. If $A = A_0B$, then $N(A/B) = N(A)/B$.*

Proof. Let $N(A/B) = N/B$. Then A is invariant in N, i.e. $N(A) \supseteq N$, and $N(A)/B \supseteq N/B = N(A/B)$. Since A is invariant in $N(A)$, the subgroup A/B is invariant in $N(A)/B$. But then $N(A/B) \supseteq N(A)/B$, and this completes the proof.

Proposition 1. *Suppose H contains a Carter subgroup of F, where F is some π-Hall subgroup of a π-solvable group G. Then H is a π-Hall subgroup in $N(H)$, if $H \subseteq F$.*

Proof. Let M be a minimal normal divisor in G. Consider the following two possibilities for M.

1) M is a π-subgroup.

Then $M \subseteq F$ and $N(HM) \cap F = HM$, by Lemma 1. Hence HM/M contains a Carter subgroup of F/M. By induction, HM/M is a π-Hall subgroup in $N(HM/M)$. By Lemma 5, $N(HM/M) = N(HM)/M$, i.e. HM is a π-Hall subgroup in $N(HM)$. But from $HM \subseteq F$ it follows that $N(H) \cap HM = H$. Therefore $N(H) \cap N(HM) = N(H)$ contains H as a π-Hall subgroup.

2) M is a π'-subgroup.

By induction, HM/M is a π-Hall subgroup in $N(HM/M) = N(HM)/M$. Therefore H is a π-Hall subgroup in $N(HM)$. Since $N(H) \subseteq N(HM)$, H is a π-Hall subgroup in $N(H)$. The proposition is proved.

Corollary 1. *Let a solvable group $G = H_1H_2$, where H_1 and H_2 are subgroups of relatively prime orders. If F_i is a Carter subgroup in H_i, $i = 1,2$, and $\{F_1, F_2\} = F_1 \times F_2$, then $F = F_1 \times F_2$ is a Carter subgroup in G.*

Indeed, the result follows from the relation $N(F) \subseteq N(F_1) \cap N(F_2)$ and Proposition 1.

Corollary 2. *Let H be a Carter subgroup of a π-Hall subgroup of a solvable group G. If $N(H)$ is a π-decomposable group, then $N(H)$ contains a Carter subgroup of the group G.*

The proof is no different from the proof of Lemma 4.

Proposition 2 generalizes the main result of the note ([3]).

Following ([1]), we shall call a subgroup H **abnormal** in G if $g \in \{H, H^g\}$ for all g in G. H is abnormal in G if and only if H does not lie in two distinct conjugate subgroups of the group G, and all subgroups containing H coincide with their normalizers in G. Lemma 6 makes it possible to shorten the proof of Proposition 2.

Lemma 6. *Let the subgroup F/M be abnormal in the group G/M. If H is such an abnormal subgroup in F that $HM = F$, then H is abnormal in G.*

Proof. Let the subgroup D contain H. Then

$$N(D) \subseteq N(DM) = N(DF) = DF.$$

From $N(D) = D \cdot [F \cap N(D)]$ and the invariance of $D \cap [F \cap N(D)] = D \cap F$ in $F \cap N(D)$, it follows that

$$N(D) = D \cdot [F \cap N(D)] = D \cdot (D \cap F) = D.$$

Now let $H \subseteq K \cap K^g$. Then $F = HM \subseteq KM \cap K^{gM}$, and therefore, from the abnormality of F in G, we obtain $K^{gM} = KM$ and g is contained in KM. Then $g = xy$ with x from K and y from M. Further, $H \subseteq K \cap K^{xy} = K \cap K^y$ and $H^{y^{-1}} \subseteq K \cap K^{y^{-1}}$. Therefore $y^{-1} \in \{H, H^{y^{-1}}\} \subseteq K$ and $K^g = K^{xy} = K$. This proves the abnormality of H in G.

Proposition 2. *A π-solvable group G contains an abnormal π-decomposable subgroup.*

Proof. M is a minimal normal divisor in G, and is an ω-subgroup, where ω is one of the sets π, π'. By induction, G/M contains a π-decomposable abnormal subgroup F/M. Let F_1 be a π-Hall subgroup and F_2 a π'-Hall subgroup in F. If $\omega = \pi'$, then put $H = N(F_1) \cap F$. If, however, $\omega = \pi$, then put $H = H_1 \times F_2$, where H_1 is a Carter subgroup in $N(F_2) \cap F_1$. From the π-solvability of F it follows that H is abnormal in F. In the first case, $HM = F$ is obtained as a result of applying the Frattini argument. In the second case the same equality follows from

$$N(HM/M) \cap F/M = HM/M,$$

the π-decomposability of F/M, and the inclusion $F_2 \subseteq HM$. Now the abnormality of H in G is a consequence of Lemma 6.

§ 3. In this paragraph we generalize Wielandt' s theorem on the solvability of the product of pairwise permutable groups of relatively prime orders to the case in which the factors are nilpotent ([5]).

Lemma 7. *Let N be a normal divisor of the group $G = AB$, $(|A|, |B|) = 1$. Then the subgroup N is factorable.*

Recall that a subgroup N of the group $G = AB$ is factorable if

$$N = (N \cap A) \cdot (N \cap B).$$

Now the lemma follows from the equality

$$|N| = |N \cap A| \cdot |N \cap B|.$$

Lemma 8. *Let $G = AB$, $(|A|, |B|) = 1$, and let A_0 and B_0 be normal divisors in A and B. Then the generated subgroup $H = \{A_0, B_0\}$ is factorable.*

By Lemma 7, from ([5]) the subgroup $F = N(H)$ is factorable, i.e.

$$F = (F \cap A) \cdot (F \cap B).$$

But then H, by Lemma 7, is factorable in F, i.e.

$$H = (H \cap F \cap A) \cdot (H \cap F \cap B) = (H \cap A) \cdot (H \cap B).$$

Lemma 9. *The group $G = \{a, b \mid a^2 = b^2 = 1\}$ is dihedral.*

Indeed,

$$G = \{a, ab\}$$

and

$$a \cdot ab \cdot a = ba = (ab)^{-1}.$$

Theorem 2. *Let $G = AB$, $(|A|, |B|) = 1$; $A = P \times L$, where P is a Sylow 2-subgroup in G, and B is nilpotent. Then the group G is solvable.*

Proof. Suppose that G is a counterexample of minimal order. Since all homomorphic images are factorized in the same way as G, application of Lemma 7 and induction shows that: (a) the group G is simple. Using (a) and considering the indices $|G : C(a)|$ and $|G : C(b)|$, where a and b are nonidentity elements of $Z(A)$ and $Z(B)$, respectively, we conclude, by the well-known theorem of Burnside [6], that: (b) $L \neq 1$ and the subgroup B is not primary. From the theorem on the solvability of groups of odd order it follows that (c) $P \neq 1$.

Let u be an involution in $Z(P)$. Suppose that $C(u) \neq A$. Since $C(u)$ is factorized, there is in $B \cap C(u)$ a nonidentity element b of prime order. By (b), in B there exists a Sylow subgroup Q whose order is relatively prime to the order of b. Then $H = \{u, Q\} \subseteq C(b) \neq G$. By Lemma 8, the subgroup H is factorized; hence H, by induction, is solvable. Let R be a nonidentity Sylow subgroup in the Fitting subgroup of the group H. If $R \subseteq H \cap A$, then $F = \{L, Q\} \subseteq N(R) \neq G$. By Lemma 8, the subgroup F is factorized; hence it is, by induction, solvable. Then, by Lemma 10 of (5), we have $L^x \cdot Q^y = Q^y \cdot L^x$ for all x and y from G. Since $LQ_0 \neq G$, G, by Theorem 3 of (7), is not simple, which contradicts (a). If, however, $R \subseteq H \cap B = Q$, then all elements conjugate to u in G lie in $\{u, B\} \subseteq N(R) \neq G$, which, by (a), is impossible. Thus: (d) if u is an involution from $Z(P)$, then $C(u) = A$.

At the same time we have proved that: (e) if u is an involution from $Z(P)$, and Q is a nonidentity Sylow subgroup in B, then $\{u, Q\} = G$.

Let u be chosen as in (d), and let v be an involution of G distinct from u. Suppose that the dihedral group (Lemma 9) $D = \{u, v\}$ is not a 2-subgroup. Then it follows from (d) that u does not lie in the center of D; therefore u inverts all elements of odd order in D. Let T be a nonidentity primary subgroup of odd order of D. From $D \subseteq N(T)$ it follows that $\{u\} \cdot T$ is a subgroup and, moreover, as we showed above, a nonnilpotent one. By the well-known D-theorem of Wielandt [8], the number $2 \cdot |T|$ does not divide the number $|A|$; hence the number $|T|$ divides the number $|B|$. Without loss of generality, we may assume that T lies in B (otherwise we would replace B by a suitable conjugate subgroup). Let Q be such a nonidentity Sylow subgroup in B that $(|T|, |Q|) = 1$ (see (b)). Then $\{u, Q\} \subseteq N(T) \neq G$, which contradicts (e). Thus: (f) if u is an involution in $Z(P)$, and v is an arbitrary involution in G, then their generated subgroup $\{u, v\}$ is a 2-subgroup.

Let u and v be the same involutions as in (f). By (f) and Sylow' s theorem, in G there is a Sylow 2-subgroup P^g that contains the subgroup $\{u, v\}$. Then $\{u, v\} \subseteq A^g$ and $L^g \subseteq C(u) = A$. This means that $L^g = L$ and $v \in N(L)$. Thus an arbitrary involution v of the group G lies in $N(L) \neq G$, which contradicts the simplicity of the group G—a contradiction that proves the theorem completely. With the aid of induction it is easy to prove

Lemma 10. Let

$$G = A_1 \cdot A_2 \cdot \ldots \cdot A_n,$$

where $(|A_i|, |A_j|) = 1$ and the products $A_{iA}j = A_{jA}i$ are solvable for all i and j. Then G is also solvable.

A consequence of Theorem 2 and Lemma 10 is

Theorem 3. Let

$$G = A_0 \cdot A_1 \cdot A_2 \cdot \ldots \cdot A_n,$$

where $(|A_i|, |A_j|) = 1$, $A_{iA}j = A_{jA}i$ for all i and j. If A_0 is 2-solvable, and the A_i, $i > 0$, are nilpotent of odd orders, then the group G is solvable.

There remains unresolved the interesting question of whether G will be solvable (G from Lemma 10) if the requirement that the numbers $|A_i|$ be relatively prime is dropped. The following is easily proved.

Theorem 4. Let $G = AB$, where A is p-solvable, and B is a Hamiltonian group of order not divisible by p. Then G is p-solvable.

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Note: Figure translations are in progress. See original paper for figures.