

ON THE CRITICAL DETERMINANT OF THE DOMAIN $\{(|x|^p + |y|^p \leq 1 \}$

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Abstract

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MATHEMATICS

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ON THE CRITICAL DETERMINANT OF THE DOMAIN $|x|^p + |y|^p \leq 1$

(Presented by Academician Yu. V. Linnik on 4 XII 1965)

Let $p \geq 1$ be a real number. Consider in the plane the domain D_p

$$|x|^p + |y|^p \leq 1. \quad (1)$$

Minkowski ⁽¹⁾ posed the problem of finding the critical determinant $\Delta(D_p)$ of the domain D_p (for the concepts from the geometry of numbers used in the present note, see, for example, ⁽²⁾). This problem reduces ^(1, 3) to the following analytic problem. Let τ_p be the real root of the equation

$$\tau_p^p + 1 = 2(1 - \tau_p)^p, \quad 0 < \tau_p < 1. \quad (2)$$

One seeks the absolute minimum of the function

$$\Delta_p(\tau, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-1/p}(1 + \sigma^p)^{-1/p} \quad (3)$$

under the condition

$$[(1 + \tau^p)^{-1/p}(1 + \sigma^p)^{-1/p}]^p + [\tau(1 + \tau^p)^{-1/p} + \sigma(1 + \sigma^p)^{-1/p}]^p = 1; \quad (4)$$

here τ and σ belong to the intervals $0 \leq \tau \leq \tau_p$, $1 \leq \sigma \leq \sigma_p$, where $\sigma_p = (2^p - 1)^{1/p}$.

As τ increases from 0 to τ_p , the number $\sigma = \sigma(\tau)$, determined by equation (4), decreases from σ_p to 1. Let $\Delta_p^{(0)} = \Delta_p(0, \sigma_p)$, $\Delta_p^{(1)} = \Delta_p(\tau_p, 1)$. Minkowski ⁽¹⁾ conjectured that $\Delta(D_p) = \min(\Delta_p^{(0)}, \Delta_p^{(1)})$. The present note is devoted to this conjecture of Minkowski. In the cases $p = 1$ and $p = 2$ Minkowski's conjecture is trivial ⁽¹⁾. For $p = 4$ it was proved by Mordell ⁽⁴⁾ with the aid of his developed theory of forms. Cohn ⁽³⁾ gave a brief sketch of a proof that Minkowski's conjecture is valid for sufficiently large p , without, however, indicating the boundary starting from which this conjecture is valid.

In the present note a method is described which makes it possible, for each particular p (except, perhaps, for some exceptional values of p), to establish whether Minkowski's conjecture is true or not. The method essentially includes the use of electronic computers. In particular, by this method the following proposition was obtained.

Theorem. *Minkowski's conjecture is valid for*

$$p = 1.3; \quad 1.4; \quad 1.5; \quad 1.6; \quad 1.7; \quad 2.2; \quad 2.3; \quad 3; \quad 4; \quad 5. \quad (5)$$

In other words,

$$\Delta(D_p) = \min(\Delta_p^{(0)}, \Delta_p^{(1)}), \quad (6)$$

if p is chosen from the set (5).

Below this theorem is proved for the case $p = 3$. The computations given below were carried out on the BESM-2 electronic computer of the Leningrad Branch of the Central Economics and Mathematics Institute of the Academy of Sciences of the USSR. For the proof a number of auxiliary propositions are used.

Lemma 1. For $p = 3$ the function

$$\Delta_p(\tau, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-1/p}(1 + \sigma^p)^{-1/p},$$

defined in the rectangle $L_p : [0, \tau_p; 1, \sigma_p]$, increases monotonically in each of its arguments when the other argument is fixed.

Indeed, for $p = 3$ we have $\tau^2\sigma \leq \tau_3^2\sigma_3 < 0.3^2 \cdot 2 = 0.18$, $\tau\sigma^2 \leq \tau_3\sigma_3^2 < 0.21 \cdot 2^2 = 0.84$, and therefore

$$\partial\Delta_p(\tau, \sigma)/\partial\tau = (1 + \tau^p)^{-1/p-1}(1 + \sigma^p)^{-1/p}(1 - \tau^{p-1}\sigma) > 0,$$

$$\partial\Delta_p(\tau, \sigma)/\partial\sigma = (1 + \tau^p)^{-1/p}(1 + \sigma^p)^{-1/p-1}(1 - \tau\sigma^{p-1}) > 0.$$

On the surface given by the function $\Delta_p(\tau, \sigma)$ in the rectangle L_p , define the level line

$$\begin{aligned} (\tau + \bar{\sigma})(1 + \tau^p)^{-1/p}(1 + \bar{\sigma}^p)^{-1/p} &= \Delta_p^{(1)} \\ \left(\Delta_3^{(1)} = \sqrt[3]{1 + \sqrt{17}/16} + \sqrt[3]{1 - \sqrt{17}/16} = 0.9529 \dots \right). \end{aligned} \quad (7)$$

We denote the resulting function by $\bar{\sigma}(\tau)$. From Lemma 1 it follows directly that for $p = 3$ the function $\bar{\sigma}(\tau)$ decreases monotonically on the interval $0 \leq \tau \leq \tau_p$.

Lemma 2. Let $p = 3$. Then

$$\sigma(\tau) > \bar{\sigma}(\tau) \quad \text{for } 0 \leq \tau \leq 0.121. \quad (8)$$

Proof. Let $0 \leq \tau' \leq \tau \leq \tau'' \leq 0.121$; by the preceding, $\bar{\sigma}(\tau) \leq \bar{\sigma}(\tau')$, $\sigma(\tau) \geq \sigma(\tau'')$. Therefore, to prove inequality (8) on the interval $\tau' \leq \tau \leq \tau''$, it is enough to prove that $\sigma(\tau'') > \bar{\sigma}(\tau')$. The interval $0 \leq \tau \leq 0.121$ was divided into subintervals $[\tau', \tau'']$ of length 0.001. The values $\sigma(\tau'')$, $\bar{\sigma}(\tau')$ at the division points were computed on a BESM-2 machine. The computations confirmed inequality (8).

Lemma 3. Let $p = 3$. Then

$$d^2\sigma/d\tau^2 - d^2\bar{\sigma}/d\tau^2 > 0 \quad (9)$$

for $\tau_p - 0.1 \leq \tau \leq \tau_p$.

Proof. Let $\sigma = \sigma(\tau)$ be defined by equation (4). Denote

$$(1 + \tau^p)^{-1/p} - (1 + \sigma^p)^{-1/p} = A, \quad \tau(1 + \tau^p)^{-1/p} + \sigma(1 + \sigma^p)^{-1/p} = B.$$

Then

$$\begin{aligned} & (1 + \sigma^p)^{-3/p-3} [B^{p-1} + \sigma^{p-1} A^{p-1}]^3 d^2\sigma/d\tau^2 \\ &= 2(p-1)(1 + \tau^p)^{-2/p-2} (1 + \sigma^p)^{-2/p-2} [B^{p-1} - \tau^{p-1} A^{p-1}] [B^{p-1} + \sigma^{p-1} A^{p-1}] \\ & \quad \times [B^{p-2} - \tau^{p-1} \sigma^{p-1} A^{p-2}] \\ & \quad + (1 + \sigma^p)^{-2/p-2} [B^{p-1} + \sigma^{p-1} A^{p-1}]^2 \left\{ (p+1)\tau^{p-1} (1 + \tau^p)^{-1/p-2} [B^{p-1} - \tau^{p-1} A^{p-1}] \right. \\ & \quad \left. - (p-1)(1 + \tau^p)^{-1/p-1} [(1 + \tau^p)^{-1/p-1} B^{p-2} - \tau^{p-2} \{(1 + \tau^p)^{-1/p-1} \right. \\ & \quad \left. - (1 + \sigma^p)^{-1/p} \} A^{p-2}] \right\} + (1 + \tau^p)^{-2/p-2} [B^{p-1} - \tau^{p-1} A^{p-1}]^2 \\ & \quad \times \left\{ (p+1)\sigma^{p-1} (1 + \sigma^p)^{-1/p-2} [B^{p-1} + \sigma^{p-1} A^{p-1}] \right. \\ & \quad \left. + (p-1)(1 + \sigma^p)^{-1/p-1} [(1 + \sigma^p)^{-1/p-1} B^{p-2} + \sigma^{p-2} \{(1 + \tau^p)^{-1/p} \right. \\ & \quad \left. - (1 + \sigma^p)^{-1/p-1} \} A^{p-2}] \right\}. \end{aligned} \quad (10)$$

By differentiating with respect to τ , taking into account the monotone decrease of the function $\sigma = \sigma(\tau)$, we verify that B , $(1 + \sigma^p)^{-3/p-3}$, $(1 + \sigma^p)^{-2/p-2}$, $(1 + \sigma^p)^{-1/p-2}$, $(1 + \sigma^p)^{-1/p-1}$ increase monotonically in τ ; A , $(1 + \tau^p)^{-1/p-1} - (1 + \sigma^p)^{-1/p}$, $(1 + \tau^p)^{-1/p} - (1 + \sigma^p)^{-1/p-1}$ decrease monotonically in τ on the interval $[0, \tau_p]$. Since $B > A$, it follows that $B^{p-1} - \tau^{p-1} A^{p-1} > 0$ for $p > 1$, and $B^{p-2} - \tau^{p-1} \sigma^{p-1} A^{p-2} > 0$ for $p > 2$. Further,

$$(1 + \tau^p)^{-1/p} - (1 + \sigma^p)^{-1/p-1} > 0,$$

$$1/2 \geq (1 + \tau^p)^{-1/p-1} - (1 + \sigma^p)^{-1/p} > 0,$$

whence it follows that, for $p > 2$,

$$(1 + \tau^p)^{-1/p-1} B^{p-2} - \tau^{p-2} \{(1 + \tau^p)^{-1/p-1} - (1 + \sigma^p)^{-1/p}\} A^{p-2} > 0.$$

Therefore, for $p > 2$, for $0 \leq \tau' \leq \tau \leq \tau'' \leq \tau_p$, the inequality

$$\begin{aligned}
 & (1 + \sigma^p)^{-3/p-3} |_{\tau'} [B^{p-1}|_{\tau'} + \sigma^{p-1}|_{\tau'} A^{p-1}|_{\tau'}]^3 d^2\sigma/d\tau^2 \\
 & > 2(p-1) \times \\
 & \quad \times (1 + \tau^p)^{-2/p-2} |_{\tau''} (1 + \sigma^p)^{-2/p-2} |_{\tau'} [B^{p-1}|_{\tau'} - \tau^{p-1} |_{\tau''} A^{p-1}|_{\tau'}] [B^{p-1}|_{\tau'} \\
 & \quad + \sigma^{p-1} |_{\tau''} A^{p-1}|_{\tau''}] [B^{p-2}|_{\tau'} - \tau^{p-1} |_{\tau''} \sigma^{p-1} |_{\tau'} A^{p-2}|_{\tau'}] \\
 & \quad + (1 + \sigma^p)^{-2/p-2} |_{\tau'} [B^{p-1}|_{\tau'} + \sigma^{p-1} |_{\tau''} A^{p-1}|_{\tau''}]^2 (p+1) \tau^{p-1} |_{\tau'} (1 + \tau^p)^{-1/p-2} |_{\tau''} \times \\
 & \quad \times [B^{p-1}|_{\tau'} - \tau^{p-1} |_{\tau''} A^{p-1}|_{\tau''}] + (1 + \tau^p)^{-2/p-2} |_{\tau''} [B^{p-1}|_{\tau'} - \tau^{p-1} |_{\tau''} A^{p-1}|_{\tau''}]^2 \times \\
 & \quad \times (p+1) \sigma^{p-1} |_{\tau''} (1 + \sigma^p)^{-1/p-2} |_{\tau'} [B^{p-1}|_{\tau'} + \sigma^{p-1} |_{\tau''} A^{p-1}|_{\tau''}] \\
 & \quad - (1 + \sigma^p)^{-2/p-2} |_{\tau''} [B^{p-1}|_{\tau''} + \sigma^{p-1} |_{\tau'} A^{p-1}|_{\tau'}]^2 (p-1) (1 + \tau^p)^{-1/p-1} |_{\tau'} \times \\
 & \quad \times [(1 + \tau^p)^{-1/p-1} |_{\tau'} B^{p-2}|_{\tau''} \\
 & \quad - \tau^{p-2} |_{\tau'} \{(1 + \tau^p)^{-1/p-1} |_{\tau''} - (1 + \sigma^p)^{-1/p} |_{\tau''}\} A^{p-2}|_{\tau''}] \\
 & \quad - (1 + \tau^p)^{-2/p-2} |_{\tau'} [B^{p-1}|_{\tau''} - \tau^{p-1} |_{\tau'} A^{p-1}|_{\tau''}]^2 (p-1) (1 + \sigma^p)^{-1/p-1} |_{\tau''} \\
 & \quad \times [(1 + \sigma^p)^{-1/p-1} |_{\tau''} B^{p-2}|_{\tau''} + \sigma^{p-2} |_{\tau'} \{(1 + \tau^p)^{-1/p} |_{\tau'} - (1 + \sigma^p)^{-1/p-1} |_{\tau'}\} A^{p-2}|_{\tau'}].
 \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned}
 & (1 + \tau^p |_{\tau'})^2 (1 - \tau |_{\tau''} \bar{\sigma}^{p-1} |_{\tau'})^3 d^2\bar{\sigma}/d\tau^2 < (1 + \bar{\sigma}^p |_{\tau'}) [(p-1) (\tau^{p-1} |_{\tau''} \\
 & \quad + \tau^{p-2} |_{\tau''} \bar{\sigma} |_{\tau'} + \bar{\sigma}^{p-1} |_{\tau'} + \tau |_{\tau''} \bar{\sigma}^{p-2} |_{\tau'}) + (p+1) (\tau^{2p-1} |_{\tau''} \bar{\sigma}^p |_{\tau'} \\
 & \quad + \tau^{p+1} |_{\tau'} \bar{\sigma}^{2p-2} |_{\tau'} + \tau^p |_{\tau''} \bar{\sigma}^{2p-1} |_{\tau'} + \tau^{2p-2} |_{\tau'} \bar{\sigma}^{p+1} |_{\tau'}) \\
 & \quad - 2(2p-1) \tau^p |_{\tau'} \bar{\sigma}^{p-1} |_{\tau''} - 2(2p-1) \tau^{p-1} |_{\tau'} \bar{\sigma}^p |_{\tau''} - 2\tau^{2p} |_{\tau'} \bar{\sigma}^{2p-1} |_{\tau''} \\
 & \quad - 2\tau^{2p-1} |_{\tau'} \bar{\sigma}^{2p} |_{\tau''}].
 \end{aligned} \tag{12}$$

The interval $[\tau_p - 0.1, \tau_p]$ was divided into subintervals $[\tau', \tau'']$ of length 0.001. On the interval $[\tau', \tau'']$, $d^2\sigma/d\tau^2$ was estimated from below on the BESM-2 with the aid of (11), while $d^2\bar{\sigma}/d\tau^2$ was estimated from above with the aid of (12). The computations confirmed inequality (9).

Lemma 4. Let $p = 3$. Then inequality (8) holds for $0 \leq \tau < \tau_p$.

Since $d\sigma/d\tau|_{\tau_p} = d\bar{\sigma}/d\tau|_{\tau_p}$, $\bar{\sigma}(\tau_p) = \sigma(\tau_p)$, it follows from Lemma 3 that inequality (8) holds in the interval $\tau_p - 0.1 \leq \tau < \tau_p$. Now Lemma 4 follows from Lemma 2.

Proof of the theorem. Let $0 \leq \tau < \tau_3$. By Lemma 4 we have $\sigma < \bar{\sigma}$. Therefore, by Lemma 1,

$$\Delta_3^{(1)} = \min(\Delta_3^{(0)}, \Delta_3^{(1)}) = \Delta_3(\tau, \bar{\sigma}) < \Delta_3(\tau, \sigma),$$

which is equivalent to the assertion of the theorem.

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Note: Figure translations are in progress. See original paper for figures.

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