SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A WEDGE HAVING A FINITE CUT

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Abstract

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SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A WEDGE HAVING A FINITE CUT

(Presented by Academician N. I. Muskhelishvili on 26 VII 1965)

Let an elastic body in the plane $\zeta = \xi + i\eta$ occupy the angle $-\alpha < \arg\zeta < \alpha,\ 0 < \alpha \leq \pi$, which is cut along the bisector from the vertex of the angle; we shall take the length of the cut to be equal to unity.

Let the boundary $\arg\zeta = \pm\alpha$ be free of external stresses (this may be assumed without loss of generality), and let the stress components on the cut be prescribed:

$$Y_y = p_1(\xi), \quad X_y = q_1(\xi) \quad \text{on the upper edge of the cut,}$$

$$Y_y = p_2(\xi), \quad X_y = q_2(\xi) \quad \text{on the lower edge of the cut,} \tag{1}$$

where $p_1(\xi), p_2(\xi), q_1(\xi), q_2(\xi)$ are absolutely continuous functions.

This problem, in the case when $\alpha = \pi/2$ (a half-plane with a cut), was solved in work ([1]). We propose another way of solving the problem.

Map the region occupied by the body onto the infinite strip $-1 \leq y \leq 1,\ -\infty < x < \infty$, cut along the negative half-axis $x < 0,\ y = 0$. The mapping is given by the function $\zeta = e^{\alpha z}$.

Consider two strips $G_1(-\infty < x < \infty,\ 0 < y < 1)$ and $G_2(-\infty < x < \infty,\ -1 < y < 0)$. By the method of N. I. Muskhelishvili ([2]), the problem formulated above is reduced to the problem of determining functions $\Phi_{01}(z), \Psi_{01}(z)$ and $\Phi_{02}(z), \Psi_{02}(z)$, holomorphic in the strips G_1 and G_2, with the following boundary conditions:

$$\Phi_{01}(z_0) + \overline{\Psi_{01}(z_0)} = \alpha\left[Y_y^{(1)} - iX_y^{(1)}\right] e^{\alpha z_0}, \quad \operatorname{Im} z_0 = 0,$$

$$\Phi_{01}(z_0) + \overline{\Psi_{01}(z_0)} + \frac{1}{\alpha}\left(e^{2i\alpha} - 1\right)\overline{\Phi_{01}(z_0)} = 0, \quad \operatorname{Im} z_0 = 1; \tag{2}$$

$$\Phi_{02}(z_0) + \overline{\Psi_{02}(z_0)} = \alpha\left[Y_y^{(2)} - iX_y^{(2)}\right]e^{\alpha z_0}, \quad \operatorname{Im} z_0 = 0;$$

$$\Phi_{02}(z) + \overline{\Psi_{02}(z_0)} + \frac{1}{\alpha}\left(e^{-2i\alpha} - 1\right)\overline{\Phi_{02}(z_0)} = 0, \quad \operatorname{Im} z_0 = -1; \tag{3}$$

$$\Phi_{01}(z) - \Phi_{02}(z_0) = \Psi_{01}(z_0) - \Psi_{02}(z_0) = 0 \quad \text{for } \operatorname{Im} z_0 = 0,\ \operatorname{Re} z_0 > 0; \tag{4}$$

$$Y_y^{(1)}(z_0) - Y_y^{(2)}(z_0) = X_y^{(1)}(z_0) - X_y^{(2)}(z_0) = 0 \quad \text{for } \operatorname{Im} z_0 = 0,\ \operatorname{Re} z_0 > 0, \tag{5}$$

where

$$Y_y^{(k)}(z_0) = Y_y\left(e^{\alpha x}\cos\alpha y,\ e^{\alpha x}\sin\alpha y\right), \quad X_y^{(k)}(z_0) = X_y\left(e^{\alpha x}\cos\alpha y,\ e^{\alpha x}\sin\alpha y\right),$$

$$(x, y) \in G_k,$$

$$\Phi_{0k}(z) = \frac{d}{dz}\varphi(e^{\alpha z}), \quad \Psi_{0k}(z) = \frac{1}{\alpha}\Phi_{0k}(z) + \frac{d}{dz}\psi(e^{\alpha z}), \quad (x, y) \in G_k,\ k = 1, 2.$$

$X_y^{(k)}, Y_y^{(k)}$ are the stress components in the strip G_k, and $\varphi(\zeta), \psi(\zeta)$ are the complex potentials for the wedge with a cut.

We shall seek the analytic functions $\Phi_{0k}(z), \Psi_{0k}(z),\ z \in G\ (k = 1, 2)$, in the following form:

$$\Phi_{0k}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{A_k(t)}{t}e^{-itz}\,dt - \frac{1}{\sqrt{2\pi}}\frac{d}{dz}\int_{-\infty}^{\infty}\frac{A_k(t)}{t}e^{-itz}\,dt - c_k, \qquad z \in G_k, \tag{6}$$

$$\Psi_{0k}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{B_k(t)}{t}e^{-itz}\,dt - \frac{1}{\sqrt{2\pi}}\frac{d}{dz}\int_{-\infty}^{\infty}\frac{B_k(t)}{t}e^{-itz}\,dt + \bar{c}_k, \qquad z \in G_k, \tag{7}$$

where

$$c_k = \lim_{x\to-\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A_k(t)}{t} e^{-itz}\, dt;$$

at the point $t = 0$ the integrals are understood in the sense of the Cauchy principal value.

Below we shall show that the desired functions $A_k(t)$, $B_k(t)$ $(k = 1, 2)$ are Fourier transforms of summable functions, continuous on the entire axis, except, possibly, at the point $t = 0$. In what follows the class of such functions will be denoted by R_0'.

In the representations (6) and (7) passage to the limit is possible both under the sign of differentiation and under the sign of integration (as z tends to the boundary G_k), and differentiation may be performed any number of times under the integral sign when $\operatorname{Im} z \neq 0$.

Let us introduce some further definitions and notation. Denote by R_0 the totality of all functions

$$\Omega(t) = \int_{-\infty}^{\infty} \omega(x) e^{itx} dx, \quad \text{where} \quad \omega(x) \in L(-\infty, \infty);$$

R_0 is a certain ring of continuous functions on the closed line ([3]). Further, denote by R_0^+ (R_0^-) the subring of R_0 composed of functions

$$\Omega^+(t) = \int_0^{\infty} \omega(x) e^{itx} dx \quad \left(\Omega^-(t) = \int_{-\infty}^{0} \omega(x) e^{itx} dx\right).$$

The ring obtained by extending R_0 (R_0^+, R_0^-) by adjoining the identity to it will be denoted by R (R^+, R^-).

Denote by $S_k(t)$, $T_k(t)$, $P_k(t)$, $Q_k(t)$ the Fourier transforms of the functions $ae^{\alpha x} Y_y^{(k)}(x, 0)$, $ae^{\alpha x} X_y^{(k)}(x, 0)$, $ae^{\alpha x} p_k(e^{\alpha x})$, $ae^{\alpha x} q_k(e^{\alpha x})$ $(k = 1, 2)$.

Substituting the representations (6) and (7) into the boundary conditions (2) and (3), performing the Fourier transform and taking into account that $S_k(t) = \overline{S_k(-t)}$, $T_k(t) = \overline{T_k(-t)}$ $(k = 1, 2)$, we obtain

$$A_1(t) = -\frac{t\left[2 \operatorname{sh} t\, e^{-t} - it\, \dfrac{e^{2i\alpha} - 1}{\alpha}\right]}{4\left[\operatorname{sh}^2 t - \left(\dfrac{\sin\alpha}{\alpha}\right)^2 t^2\right](1 + it)} S_1(t) + \frac{t\left[2 \operatorname{sh} t\, e^{-t} + it\, \dfrac{e^{2i\alpha} - 1}{\alpha}\right]}{4\left[\operatorname{sh}^2 t - \left(\dfrac{\sin\alpha}{\alpha}\right)^2 t^2\right](1 + it)} iT_1(t), \tag{8}$$

$$A_2(t) = \frac{t\left[2\operatorname{sh} t\, e^t + it\,\dfrac{e^{-2i\alpha}-1}{\alpha}\right]}{4\left[\operatorname{sh}^2 t - \left(\dfrac{\sin\alpha}{\alpha}\right)^2 t^2\right](1+it)} S_2(t) - \frac{t\left[2\operatorname{sh} t\, e^t - it\,\dfrac{e^{-2i\alpha}-1}{\alpha}\right]}{4\left[\operatorname{sh}^2 t - \left(\dfrac{\sin\alpha}{\alpha}\right)^2 t^2\right](1+it)} iT_2(t); \tag{9}$$

$$B_k(t) = \frac{t}{1+it}\left[S_k(t) + iT_k(t)\right] + \bar{A}_k(-t) \qquad (k = 1, 2). \tag{10}$$

By virtue of condition (4) we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{A_1(t)-A_2(t)}{t}e^{-itx}\,dt - \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{A_1(t)-A_2(t)}{t}e^{-itx}\,dt + c_2 - c_1 = 0$$

for $x > 0$,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{B_1(t)-B_2(t)}{t}e^{-itx}\,dt - \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{B_1(t)-B_2(t)}{t}e^{-itx}\,dt + \bar{c}_1 - \bar{c}_2$$

for $x > 0$;

whence we easily conclude that if $A_k(t), B_k(t) \in R_0$ $(k = 1, 2)$, then $A_1(t) - A_2(t) \in R_0^-$, $B_1(t) - B_2(t) \in R_0^-$.

Applying condition (5), we obtain

$$T_1(t) = T^+(t) + Q_1^-(t), \qquad T_2(t) = T^+(t) + Q_2^-(t); \tag{11}$$

$$S_1(t) = S^+(t) + P_1^-(t), \qquad S_2(t) = S^+(t) + P_2^-(t), \tag{12}$$

where $S^+(t)$, $T^+(t)$ are the sought functions from the class R_0^+.

From conditions (8)–(10) we shall have

$$\begin{aligned} A_1(t) - A_2(t) = &-\frac{t\left[\operatorname{sh} t\,\operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha}\,t\right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin\alpha}{\alpha}\right)^2 t\right](1+it)} S^+(t) \\ &+ \frac{t\left[\operatorname{sh} t\,\operatorname{ch} t - \frac{\sin 2\alpha}{2\alpha}\,t\right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin\alpha}{\alpha}\right)^2 t^2\right](1+it)} iT^+(t) + f_1(t); \end{aligned} \tag{13}$$

$$B_1(t) - B_2(t) = -\frac{t\left[\operatorname{sh} t\, \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha}\, t\right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin\alpha}{\alpha}\right)^2 t^2\right](1+it)} S^+(t) - \frac{t\left[\operatorname{sh} t\, \operatorname{ch} t - \frac{\sin 2\alpha}{2\alpha}\, t\right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin\alpha}{\alpha}\right)^2 t^2\right](1+it)} iT^+(t) + f_2(t), \tag{14}$$

where $f_1(t)$, $f_2(t)$ are prescribed functions.

Adding equalities (12) and (13) and introducing the notation

$$\Phi^-(t) = [A_1(t) - A_2(t) + B_1(t) - B_2(t)]\sqrt{1+it}; \tag{15}$$

$$\Phi^+(t) = -2S^+(t)\sqrt{1-it}, \tag{16}$$

we obtain

$$\Phi^+(t) = \frac{\left[\operatorname{sh}^2 t - \left(\frac{\sin\alpha}{\alpha}\right)^2 t^2\right]\sqrt{1+t^2}}{\left(\operatorname{sh} t\, \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha}\, t\right) t} \Phi^-(t) + g(t). \tag{17}$$

It is easy to see that the free term $g(t) \in R_0'$, while the coefficient of the Hilbert boundary-value problem (17) belongs to the ring of functions R. Since the coefficient is positive on the entire axis $-\infty < t < \infty$, the index of the boundary-value problem (17) is $\chi = 0$.

On the basis of work [4], the Hilbert boundary-value problem (17) in the class of functions $\Phi^\pm(t) \in R_0^\pm$ has the unique solution

$$\Phi^\pm(t_0) = \frac{X^\pm(t_0)}{2}\left[\pm\frac{g(t_0)}{X^+(t_0)} + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{g(t)\,dt}{X^+(t)(t-t_0)}\right], \tag{18}$$

where

$$X^\pm(t_0) = \exp\frac{1}{2}\left[\pm\ln\frac{\left(\operatorname{sh}^2 t_0 - \left(\frac{\sin\alpha}{\alpha}\right)^2 t_0^2\right)\sqrt{1+t_0^2}}{t_0\left(\operatorname{sh} t\, \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha}\, t_0\right)} + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln\frac{\left(\operatorname{sh}^2 t - \left(\frac{\sin\alpha}{\alpha}\right)^2 t^2\right)\sqrt{1+t^2}}{t\left(\operatorname{sh} t\, \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha}\, t\right)}}{t-t_0}\,dt\right]. \tag{19}$$

From equality (16) we have

$$S^+(t) = -\frac{1}{2\sqrt{1-it}}\,\Phi^+(t).$$

Subtracting equality (14) from equality (13), we obtain a boundary-value problem analogous to problem (17), from which $T^+(t)$ is determined. From equalities (11)–(12) it follows that $S_k(t)\sqrt{1-it} \in R_0$, $T_k(t)\sqrt{1-it} \in R_0$ $(k = 1, 2)$. Substituting the found $S_k(t)$, $T_k(t)$ $(k = 1, 2)$ into (8)–(10), we determine $A_k(t)$, $B_k(t)$ $(k = 1, 2)$. Thus, the problem is solved.

Now let us show that $A_k(t)$, $B_k(t) \in R_0'$. To this end we add to both sides of equalities (8) and (9), respectively, expressions of the form

$$\frac{te^{-t}}{(1+it)\,2\,\mathrm{ch}\,t}\,[S_1(t) - iT_1(t)], \qquad -\frac{te^{t}}{(1+it)\,2\,\mathrm{ch}\,t}\,[S_2(t) - iT_2(t)]. \tag{20}$$

It is easy to see that the right-hand sides of the resulting equalities belong to the class R_0'. In order to prove that $A_k(t)$, $B_k(t)$ belong to the class R_0', it is enough to show that the expressions (20) belong to the class R_0'. This follows from the following lemma.

Lemma. Let $f(t + i\tau)$ be an analytic function in the strip $-h < \tau < h$, satisfying the conditions $|f(t + i\tau)| < A/|t|^\beta$, $|f'(t + i\tau)| < B/|t|^\gamma$ as $|t| \to \infty$, $\beta > 0$, $\gamma > 1$; then $f(t) \in R_0'$.

Proof. The function $f(s)e^{-isx}$, $s = t + i\tau$, for $x \le 0$ is analytic in the strip $0 \le \tau < h_1$ $(h_1 < h)$ and vanishes at infinity. The same is true for $x \ge 0$ in the strip $-h_1 < \tau \le 0$. Therefore, using Cauchy' s theorem and the Sokhotski-Plemelj formulas, we obtain

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t} e^{-itx}\,dt = \begin{cases} e^{h_1 x}\Phi_1(x) + f(0)/2, \\ e^{-h_1 x}\Phi_2(x) - f(0)/2, \end{cases} \tag{21}$$

where

$$\Phi_1(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t+ih_1)}{t+ih_1} e^{-itx}\,dt, \qquad \Phi_2(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t-ih_1)}{t-ih_1} e^{-itx}\,dt. \tag{22}$$

According to the conditions of the lemma, $\Phi_1(x)$, $\Phi_2(x) \in H$ (H is the class of Hölder functions) and $\Phi_1(0) - \Phi_2(0) = f(0)$. Hence it follows that both parts of equality (21) satisfy the Hölder conditions. Let us show that $\Phi_1(x)$ and $\Phi_2(x)$ have everywhere a continuous derivative, except possibly at the point $x = 0$.

We rewrite the first expression in equality (22) as follows:

$$x\Phi_1(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{f(t+ih_1)}{t+ih_1}\right)' e^{-itx}\,dt,$$

whence, by differentiation, we obtain

$$x\Phi_1'(x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} t\left(\frac{f(t+ih_1)}{t+ih_1}\right)' e^{-itx}\,dt - \Phi_1(x).$$

According to the conditions of the lemma, the right-hand side of the last equality belongs to the class H and is equal to zero for $x = 0$. Therefore $\Phi_1'(x)$ is a continuous function for $x \neq 0$ and $|\Phi_1'(x)| < c/|x|^{\delta}$, as $x \to 0$, $\delta < 1$. It is clear that $\Phi_2'(x)$ will have the same properties. Then

$$\mu(x) \equiv \frac{d}{dx}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t}e^{-itx}\,dt\right] \in L(-\infty, \infty),$$

whence, using the inversion formula, we obtain

$$f(t) = \int_{-\infty}^{\infty}\mu(x)e^{itx}\,dx.$$

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Note: Figure translations are in progress. See original paper for figures.