ON THE LOCATION IN THE COMPLEX PLANE OF THE ZEROS OF THE FOURIER TRANSFORM

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Abstract

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MATHEMATICS

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ON THE LOCATION IN THE COMPLEX PLANE OF THE ZEROS OF THE FOURIER TRANSFORM

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In this note the distribution of the zeros of functions

$$f(z) = \int_{\sigma}^{\sigma+T} \varphi(t) e^{itz}\, dt, \tag{1}$$

where $\varphi(t)$ is continuous and $\varphi(\sigma) \neq 0$, is studied.

Theorem 1. *If $\varphi(t)$ is continuous at $t = \sigma$ and of bounded variation on the segment $[\sigma, \sigma + T]$, then some half-plane $y \geq h$ is free of zeros of the function $f(z)$.*

Proof. Integrating by parts, we obtain, as $y \to \infty$, uniformly in x,

$$f(z) = -iz^{-1} e^{i\sigma z}[1 + o(1)].$$

This asymptotic equality implies Theorem 1.

Yu. I. Lyubich posed the following question: is it possible that, for a continuous function $\varphi(t)$, $\varphi(\sigma) \neq 0$, the function $f(z)$ defined in (1) should have zeros with arbitrarily large imaginary parts? The following theorem answers this question.

Theorem 2. *Let $\psi(\delta)$, $0 < \delta < \infty$, be a function satisfying the conditions: a) $\psi(\delta) > 0$ if $\delta > 0$; b) $\lim_{\delta\to 0} \psi(\delta) = 0$, c) $\lim_{\delta\to 0} \delta^{-1}\psi(\delta) = \infty$; d) $\psi(\delta)$ increases, and $\delta^{-1}\psi(\delta)$ decreases, as δ increases.*

Then there exists a function $\varphi(t)$, satisfying the condition $\varphi(-\pi) \neq 0$, with modulus of continuity

$$\omega_\varphi(\delta) \leq \psi(\delta)$$

and such that the entire function

$$f(z) = \int_{-\pi}^{\pi} \varphi(t) e^{itz}\, dt \tag{2}$$

has an infinite set of zeros $z_k = x_k + iy_k$, for which the real and imaginary parts are related by

$$y_k = |x_k| \psi(|x_k^{-1}|) + o(1).$$

Proof. We shall construct the function $\varphi(t)$ in the form of an absolutely convergent lacunary Fourier series

$$\varphi(t) = \frac{1}{20} \sum_{k=0}^{\infty} (-1)^{n_k} c_k e^{-in_k t}, \tag{3}$$

* An analogous fact was noted earlier by T. M. Karaseva in the paper (5), § 3. In that paper the restrictions on $\varphi(t)$ are stronger. An analogous theorem from the theory of almost-periodic functions is also well known.

where n_k are integers. Substituting (3) into (2) and integrating term by term, we obtain

$$f(z) = \frac{1}{10} \sum_{k=0}^{\infty} \frac{\sin z}{z - n_k}. \tag{4}$$

The series in (4) converges absolutely and uniformly on every compact set in the complex plane.

We shall construct the numbers c_k and n_k inductively. Put $c_0 = 1$, $n_0 = 1$. Suppose that the numbers $c_0, c_1, \dots, c_k$ and $n_0 < n_1 < \cdots < n_k$ have already been constructed, and that the following conditions are satisfied:

1. $|c_l| < 2\psi(n_l^{-1}) < 2^{-l+1}(l+1)^{-1} \quad (l = 1, 2, \dots, k).$
2. $$\sum_{s=l+1}^{k} |c_s| < (2^{-1} - 2^{-(k-l+1)})|c_l| \quad (l = 0, 1, \dots, k-1).$$
3. $$\sum_{s=0}^{l-1} n_s |c_s| < n_l |c_l| \quad (l = 1, 2, \dots, k).$$
4. In each disk

$$K_s = \{z : \ |z - n_s(1 + i\psi(n_s^{-1}))| < 2^{-s-1}\}$$

there is a zero of the function

$$f_l(z) = \frac{1}{10}\sin z \sum_{j=0}^{l} c_j (z-n_j)^{-1} \quad (s=1,2,\dots,l;\ l=1,2,\dots,k).$$

Put

$$c_{k+1} = -i\psi(n_{k+1}^{-1})[1+i\psi(n_{k+1}^{-1})]\sum_{s=0}^{k} c_s \tag{5}$$

and choose n_{k+1}. By virtue of properties b) and c) of the function $\psi(\delta)$ and our choice of c_{k+1}, there exists N_1 such that, for $n_{k+1} > N_1$, conditions 1, 2, 3 of the inductive construction will be satisfied with k replaced by $k+1$. Using condition 1 with $l = k+1$, property b) of the function $\psi(\delta)$, and Rouché' s theorem, we obtain that there exists $N > N_1$ such that, for $n_{k+1} > N$, the function $f_{k+1}(z)$ will have a zero $z_{k+1,s}$ in the disk K_s $(s = 1, 2, \dots, k)$. Choose $n_{k+1} > N$ so that the inequalities

$$\psi(n_{k+1}^{-1}) < 2^{-k-6}\left(\sum_{s=0}^{k} n_s\right)^{-1}, \tag{6}$$

$$\left| f_k(z) - \frac{1}{10}\frac{\sin z}{z}\sum_{s=0}^{k} c_s \right| \le \frac{1}{5}\frac{|\sin z|}{|z|^2}\sum_{s=0}^{k} n_s \quad (2|z| > n_{k+1}). \tag{7}$$

Then the function $f_{k+1}(z)$ will have a zero $z_{k+1,k+1}$ in the disk K_{k+1}. Indeed, on the circle

$$|z - n_{k+1}(1+i\psi(n_{k+1}^{-1}))| = 2^{-k-2}$$

the inequality

$$\left|\sum_{s=0}^{k} c_s z^{-1} + c_{k+1}(z-n_{k+1})^{-1}\right| > 2^{-k-5} n_{k+1}^{-2}\psi^{-1}(n_{k+1}^{-1}) \tag{8}$$

holds, and the function

$$z^{-1}\sum_{s=0}^{k} c_s + (z-n_{k+1})^{-1}c_{k+1}$$

vanishes at the point

$$z = n_{k+1}(1+i\psi(n_{k+1}^{-1})).$$

From (9), (10), and (11), by Rouché' s theorem it follows that the function $f_{k+1}(z)$ has a zero $z_{k+1,k+1}$ in the disk K_{k+1}, and thus we have satisfied condition

4 of the inductive construction with k replaced by $k+1$. This completes the inductive construction.

By virtue of this construction the function (4) is defined, and there exists a sequence $\{z_l\}_{l=1}^{\infty}$ of its zeros satisfying the condition

$$|z_l - n_l(1 + i\psi(n_l^{-1}))| \leq 2^{-l-1}.$$

The coefficients c_l satisfy conditions 1, 2, and 3 with $k = \infty$. The condition $\varphi(-\pi) \neq 0$ is fulfilled, since, by condition 1,

$$20|\varphi(-\pi)| > |c_0| - \sum_{k=1}^{\infty} |c_k| > 1 - \sum_{k=1}^{\infty} (k+1)^{-1} \cdot 2^{-k} > 2^{-1}.$$

Let us estimate the modulus of continuity $\omega_\varphi(\delta)$ of the function $\varphi(t)$. Let m be a natural number such that $n_k\delta < 1$ for $k \leq m$ and $n_k\delta > 1$ for $k > m$. From conditions 1, 2, and 3, used with $k = \infty$, we obtain

$$\omega_\varphi(\delta) \ll \frac{1}{10} \sum_{k=1}^{m} |c_k| \cdot |e^{-in_k\delta} - 1| + \frac{2}{10} \sum_{k=m+1}^{\infty} |c_k| \leq \frac{\delta}{10} \sum_{k=1}^{m} |c_k| n_k +$$

$$+ \frac{2}{10} \sum_{k=m+1}^{\infty} |c_k| \leq \frac{2}{10} \delta |c_m| n_m + \frac{3}{10} |c_{m+1}| \ll \frac{2}{5} n_m \psi(n_m^{-1}) + \frac{3}{10} \psi(n_{m+1}^{-1}).$$

Since $n_{m+1}^{-1} < \delta < n_m^{-1}$, it follows from condition d) of Theorem 2 that

$$\omega_\varphi(\delta) \ll \psi(\delta).$$

Theorem 2 is proved.

This theorem is in a certain sense sharp, as is shown by

Theorem 3. *The function $f(z)$, defined in (1), has no zeros in the region*

$$y > C|\varphi(\sigma)|^{-1}|x|\omega_\varphi(|x|^{-1}) + C_1,$$

where C is an absolute constant; C_1 depends on $\varphi(t)$, and $\omega_\varphi(\delta)$ is the modulus of continuity of the function $\varphi(t)$.

The following theorem shows that, for any smoothness of the function $\varphi(t)$ near the left endpoint of the segment $[\sigma, \sigma + T]$, the function $f(z)$, defined in (1), may have zeros with arbitrarily large imaginary part.

Theorem 4. *Let the function $u(x)$, $0 < x < \infty$, satisfy the condition $\lim_{x\to\infty} u(x) = 0$, $0 < T_0 < T < \infty$, and let $h(x)$ be a solution of the equation*

$$x^{-1}h(x)\exp[T_0 h(x)] = u(x).$$

Then there exists a function $\varphi(t)$, continuous on $[\sigma, \sigma + T]$, satisfying the condition $\varphi(t) = \varphi(\sigma) \neq 0$, $\sigma \leq t \leq T_0 + \delta$, and such that the function $f(z)$ (1) has a sequence of zeros approaching without bound the curve $y = h(x)$.

The sharpness of Theorem 4 is shown by

Theorem 5. *If $\varphi(t)$ is continuous on the segment $[\sigma, \sigma + T]$ and of bounded variation on the segment $[\sigma, \sigma + T_0]$, $0 < T_0 < T < \infty$, $\varphi(\sigma) \neq 0$, then there exists a function $h(t)$ satisfying the condition*

$$\lim_{x\to\infty} x^{-1}h(x) \cdot \exp[T_0 h(x)] = 0$$

such that the region $y > h(x)$ is free of zeros of the function $f(z)$, defined by equality (1).

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Note: Figure translations are in progress. See original paper for figures.