



Soviet-era science, translated into English

TOPOLOGICAL EMBEDDINGS AND PSEUDOISOTOPY

MATHEMATICS

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.33924>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.833

MATHEMATICS

Lyudmila KELDYSH

TOPOLOGICAL EMBEDDINGS AND PSEUDOISOTOPY

(Presented by Academician P. S. Aleksandrov, December 7, 1965)

1. We study here arbitrary topological embeddings, i.e. homeomorphic mappings, of a two-dimensional manifold M^2 into three-dimensional Euclidean space E^3 , and embeddings of a polyhedron P^k of dimension $k \leq n/2 - 1$ into Euclidean space E^n . A topological polyhedron \mathcal{P} in E^n is called **tame** if there exists a homeomorphic mapping H of the space E^n onto itself such that $H(\mathcal{P})$ is a rectilinear polyhedron. Otherwise the polyhedron \mathcal{P} is called **wild**. A polyhedron $\mathcal{P} \subset E^n$ is called **locally tame** at a point x if there exist a neighborhood U of the point x and a homeomorphic mapping $\varphi : U \rightarrow E^n$ such that $\varphi(U \cap \mathcal{P})$ is a rectilinear polyhedron. If for a point $x \in \mathcal{P}$ no such neighborhood exists, then x is called a **point of wildness** for \mathcal{P} . Bing ⁽²⁾ and Moise ⁽⁸⁾ showed that a locally tame polyhedron in E^3 is tame. If S^2 is a locally tame topological sphere in E^3 , then there exists a homeomorphic mapping $H : E^3 \rightarrow E^3$ carrying S^2 into a polyhedral sphere and, by Aleksandrov's theorem ⁽¹⁾, onto the boundary σ^2 of a three-dimensional simplex: $H(\sigma^2) = S^2$. The homeomorphism H can be chosen so that it does not change orientation and, consequently, it can be joined by an isotopy F_t , $0 \leq t \leq 1$, with the identity mapping of E^3 ⁽⁹⁾: $F_0 = 1$, $F_1 = H$. G. Gluck ⁽⁶⁾ proved that for a locally tame polyhedron \mathcal{P}^k of dimension $k \leq n/2 - 1$ in E^n there exists an isotopy of E^n from the identity mapping carrying \mathcal{P}^k into a rectilinear polyhedron. A. V. Chernavskii proved an analogous theorem for polyhedra of dimension $k < 2/3n - 1$ ⁽¹⁰⁾.

For wildly embedded polyhedra in E^n , by the very definition, there exists no homeomorphism of E^n onto itself carrying the given polyhedron into a rectilinear one.

The following questions arise:

- 1) Does there exist, for a topological polyhedron \mathcal{P}^k in E^n , a not necessarily one-to-one mapping F of the space E^n onto itself and a rectilinear polyhedron P^k such that $F|_{P^k}$ is a homeomorphism and $F(P^k) = \mathcal{P}^k$?
- 2) Can the mapping F be chosen so that it is pseudoisotopic to the identity?

Lenning gave a positive answer to question 1) for an embedding of the two-dimensional sphere S^2 in E^3 ⁽¹¹⁾.

We give an affirmative answer to both questions for an embedding of a compact two-dimensional manifold in E^3 and for a topological polyhedron of dimension $k \leq 1/2 n - 1$ in E^n .

A **pseudoisotopy** is a homotopy F_t , $0 \leq t \leq 1$, which for every $t < 1$ is a homeomorphic mapping of the space onto itself. F_t is called an ε -**pseudoisotopy** if for any x, t and t' , $\rho[F_t(x)F_{t'}(x)] < \varepsilon$, where ρ is distance.

Theorem 1. *For an arbitrarily embedded two-dimensional sphere S^2 in E^3 there exists a pseudoisotopy from the identity mapping $F_t : E^3 \rightarrow E^3$, carrying the boundary σ^2 of a three-dimensional simplex into S^2 : $F_0 = 1$, $F_1(\sigma^2) = S^2$, such that $F_1|_{\sigma^2}$ and $F_1|_{E^3 \setminus F_1^{-1}(S^2)}$ are homeomorphisms and the set of points x whose inverse images $F_1^{-1}(x)$ are degenerate is zero-dimensional and is contained in the set of points of wildness of S^2 .*

Theorem 2. For an arbitrarily embedded two-dimensional compact manifold $M^2 \subset E^3$ and a number $\varepsilon > 0$, there exists a polyhedral manifold \mathfrak{M}^2 and an ε -pseudoisotopy F_t carrying \mathfrak{M}^2 onto M^2 and satisfying the conditions of Theorem 1.

Theorem 3. For an arbitrarily embedded in Euclidean space E^n polyhedron \mathfrak{P}^k of dimension $k \leq n/2 - 1$ and a rectilinear polyhedron $P^k \subset E^n$, homeomorphic to \mathfrak{P}^k , there exists a pseudoisotopy Φ_t from the identity map carrying P^k onto \mathfrak{P}^k : $\Phi_0 = 1$, $\Phi(P^k) = \mathfrak{P}^k$, and moreover $\Phi_1|_{P^k}$ and $\Phi_1|_{E^n \setminus \Phi_1^{-1}(\mathfrak{P}^k)}$ are homeomorphisms, and the set of points x whose inverse images $\Phi_1^{-1}(x)$ are degenerate is contained in the set of points of wildness for \mathfrak{P}^k .

Corollary. For an arbitrarily embedded in E^n polyhedron \mathfrak{P}^k of dimension $k \leq n/2 - 1$ and a number $\varepsilon > 0$, there exists a rectilinear polyhedron $P^k \subset E^n$ and an ε -pseudoisotopy carrying P^k onto \mathfrak{P}^k and satisfying the conditions of Theorem 2.

We describe here the idea of the proof of Theorem 1. The proof of Theorem 2 is analogous. The proof of Theorem 3 is based on Gluck' s Theorems 1.2 and 8.2 ⁽⁶⁾. The corollary to Theorem 3 follows from the proof of this theorem.

2. Idea of the proof of Theorem 1.

By Bing's approximation theorem, for every $\varepsilon > 0$ there exists a homeomorphic ε -shift h_ε of the sphere S^2 onto a polyhedral sphere Σ^2 . Let σ^2 be the boundary of a three-dimensional simplex lying in E^3 , and let f be a homeomorphic mapping of σ^2 onto S^2 , chosen so that for sufficiently small ε , $h_\varepsilon f : \sigma^2 \rightarrow \Sigma^2$ can be extended to an orientation-preserving homeomorphism $\varphi : E^3 \rightarrow E^3$. The isotopy F_t is constructed so that $F_1|_{\sigma^2} \equiv f$.

Choosing a decreasing sequence of numbers $\varepsilon_n \rightarrow 0$, we construct a sequence of tame spheres S_n converging to S^2 in such a way that the sphere S_{n-1} is carried by

an ε_{n-1} -isotopy $H_t^n : E^3 \rightarrow E^3$ onto S_n , and the sequence of homeomorphisms $\Phi_1^n = H_1^n H_1^{n-1} \dots H_1^1$, where $H_1^1(\sigma^2) = S_1$, converges uniformly to a continuous map $F_1 : E^3 \rightarrow E^3$ satisfying the conditions of the theorem.

By Bing's theorems ^(4,5), on the sphere S^2 one can choose an increasing sequence of tame Sierpiński curves* $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$ such that the diameters of the components of $S^2 \setminus X_n$ are less than $\varepsilon_n/2$, $X_{n+1} \setminus X_n$ lies in a finite number of components of $S^2 \setminus X_n$, and $\overline{X_{n+1} \setminus X_n}$ ** contains the boundaries of these components.

Let $\{\delta_{ni}\}$ be the open disks that are the components of $S^2 \setminus X_n$. If the sphere S^2 is wild, then such disks exist (Theorem 8.2 in ⁽⁵⁾), since X_n is tame. For each δ_{ni} we choose in E^3 a neighborhood u_{ni} such that

$$S^2 \cap u_{ni} = \delta_{ni}, \quad \overline{u_{ni}} \cap \overline{u_{nj}} = \Lambda, \quad i \neq j; \quad \text{diam } u_{ni} < 2 \text{ diam } \delta_{ni} < \varepsilon_n; \quad \overline{u_{ni}} \subset \overline{u_{n-1,j}},$$

if $\overline{\delta_{ni}} \subset \delta_{n-1,j}$; $u_{ni} = u_{n-1,j}$, if $\delta_{ni} = \delta_{n-1,j}$.

By Bing's theorem ⁽³⁾, for each disk δ_{ni} there exists a homeomorphic shift g_{ni} , fixed on the boundary $-\partial\delta_{ni}$, into a disk $\Delta_{ni} \subset u_{ni}$, locally polyhedral mod $\partial\Delta_{ni}$. We choose the homeomorphisms g_{ni} so that $g_{ni} = g_{n-1,j}$, if $\delta_{ni} = \delta_{n-1,j}$, and set

$$S_n = S^2 \setminus \bigcup \delta_{ni} \cup \bigcup \Delta_{ni}; \quad \Delta_{ni} \subset u_{ni}; \quad (1)$$

$$g_n : S^2 \rightarrow S_n; \quad g_n|_{\delta_{ni}} = g_{ni}, \quad g_n|_{S^2 \setminus \bigcup \delta_{ni}} = 1.$$

Each sphere S_n is locally tame mod X_n and, by Theorem 8.2 ⁽⁵⁾, is tame. Then $g_1 f : \sigma^2 \rightarrow S_1$, by the choice of f , can be extended to a homeo-

* A Sierpiński curve is a continuum X obtained from the sphere by deleting a countable everywhere dense set of open disks whose boundaries do not intersect. X is called tame if there exists a homeomorphism $\psi : E^3 \rightarrow E^3$ such that $\psi(X)$ lies in a plane.

** By \overline{A} we denote the closure of the set A .

morphism $H_1^1 : E^3 \rightarrow E^3$ which does not change orientation, and, consequently, there exists an isotopy $H_t^1 : E^3 \rightarrow E^3$ such that

$$H_0^1 = 1; \quad H_1^1|_{\sigma^2} = g_1 f. \quad (2)$$

The isotopies $\Phi_t^n : E^3 \rightarrow E^3$ such that $\Phi_1^n|_{\sigma^2} = g_n f$ are constructed by induction. Φ_t^n is the product of the isotopies Φ_t^{n-1} and $\widetilde{H}_t^n = H_t^n \Phi_1^{n-1}$, $\Phi_t^n = \Phi_t^{n-1} \circ \widetilde{H}_t^n$, where H_t^n , $n \geq 2$, is an ε_{n-1} -isotopy and $H_1^n|_{S_{n-1}} = g_n^{-1} g_{n-1}$. By construction, the sphere S_{n+1} differs from S_n in a finite number of regions u_{ni} , $i = 1, 2, \dots, k_n$, and

$$g_{n+1} \Big|_{S^2 \setminus \bigcup_{i=1}^{k_n} \delta_{ni}} = g_n. \tag{3}$$

For each δ_{ni} , $n = 1, 2, \dots$; $i = 1, 2, \dots, k_n$, a disk $\delta'_{ni} \subset S^2$ is chosen so that $\partial\delta'_{ni} \subset X_n$ and $\partial\delta'_{ni}$ does not meet the boundaries of the components of $S^2 \setminus X_n$, and

$$\delta'_{ni} \supset \delta_{ni}; \quad \delta'_{ni} \cap \delta'_{nj} = \Lambda, \quad i \neq j;$$

$$\text{diam } \delta'_{ni} < (1 + \varepsilon_n) \text{diam } \delta_{ni};$$

for $k < n$: $\bar{\delta}'_{nj} \subset \delta'_{ki}$, if $\bar{\delta}_{nj} \subset \delta'_{ki}$; $\bar{\delta}'_{nj} \cap \bar{\delta}'_{ki} = \Lambda$, if $\bar{\delta}_{nj} \cap \bar{\delta}_{ki} = \Lambda$.

For each disk δ'_{ni} a neighborhood u'_{ni} is constructed satisfying the conditions:

$$u'_{ni} \cap S^2 = \delta'_{ni}; \quad \text{diam } u'_{ni} < 2 \text{diam } \delta_{ni} < \varepsilon_n;$$

$$\bar{u}_{nr} \subset u'_{ni}, \text{ if } \delta_{nr} \subset \delta'_{ni}; \quad \bar{u}'_{ni} \cap \bar{u}'_{kj} = \Lambda, \text{ if } \delta'_{ni} \cap \delta'_{kj} = \Lambda \tag{4}$$

$$\bar{u}'_{ni} \subset \bar{u}'_{kr}, \text{ if } k < n \text{ and } \delta_{ni} \subset \delta'_{kr}.$$

We construct the isotopy H_t^{n+1} , carrying S_n into S_{n+1} , so that

$$H_t^{n+1} \Big|_{E^3 \setminus \bigcup_{i=1}^{k_n} u'_{ni}} = 1, \quad 0 \leq t \leq 1.$$

The construction is carried out inside the regions u'_{ni} , and we omit the indices: $\Delta = g_n(\delta) \subset u \subset u'$. In $\Delta' = S_n \cap u'$ choose a disk $\bar{\Delta}$ with boundary in X_n : $\bar{\Delta} \subset \Delta \subset \Delta'$.

$\bar{\Delta}$ is tame, since S_n is tame, and one can construct an isotopy φ_t satisfying the conditions

$$\varphi_0 = 1; \quad \varphi_t|_{\partial u'} = 1; \quad \varphi_1(\bar{\Delta}) = \tilde{d},$$

where \tilde{d} is a polyhedral disk. In view of (4) and (1), $S_{n+1} \cap u = g_{n+1}(S^2 \cap u) = D$; $S_{n+1} \cap u' = g_{n+1}(S^2 \cap u')$. $\varphi_1(\bar{\Delta} \setminus \Delta \cup D)$ is a tame disk lying in u' and locally polyhedral at the points of its boundary, since $\bar{\Delta} \setminus \Delta \cup D \subset S_{n+1}$.

Applying the theorems of Bing (2) and Kister (7), we construct an isotopy $\theta_t : u' \rightarrow u'$ such that

$$\theta_0 = 1; \quad \theta_t|_{\partial u' \cup \varphi_1(\Delta' \setminus \bar{\Delta})} = 1 \tag{5}$$

and $\tilde{D} = \theta_1 \varphi_1(\bar{\Delta} \setminus \Delta \cup D)$ is a polyhedral disk.

Then a piecewise-linear isotopy $h_t : \bar{u}' \rightarrow \bar{u}'$ is constructed so that

$$h_0 = 1; \quad h_t|_{\partial u' \cup \varphi_1(\Delta' \setminus \tilde{\Delta})} = 1; \quad h_1 \varphi_1(\tilde{\Delta}) = \tilde{D} \tag{6}$$

Denoting by φ_t^n , θ_t^n , and h_t^n the isotopies constructed as described in each u'_{ni} , $i = 1, 2, \dots, k_n$, and fixed on $E^3 \setminus \bigcup_{i=1}^{k_n} u'_{ni}$, we put

$$\Gamma^{n+1} = \varphi_t^n \circ \tilde{h}_t^n \circ \tilde{\theta}_t^n \circ \varphi_t^n,$$

where

$$\tilde{h}_t^n = h_t^n \varphi_1^n; \quad \tilde{\theta}_t^n = (\theta_t^n)^{-1} \tilde{h}_1^n; \quad \tilde{\varphi}_t^n = (\varphi_t^n)^{-1} \tilde{\theta}_1^n.$$

Γ_1^{n+1} carries each disk $\tilde{\Delta}_{ni}$ into the disk $\tilde{\Delta}_{ni} \setminus \Delta_{ni} \cup D_{ni}$ and, by (5), (6), $\Gamma_1^{n+1}|_{S_n \setminus \bigcup \tilde{\Delta}_{ni}} = 1$, hence $\Gamma_1^{n+1}(S_n) = S_{n+1}$.

The homeomorphic mapping

$$\psi^{n+1} = g_{n+1} g_n^{-1} (\Gamma_1^{n+1})^{-1} : S_{n+1} \rightarrow S_{n+1}$$

is fixed on

$$S_{n+1} \setminus \bigcup_{i=1}^{k_n} (\tilde{\Delta}_{ni} \setminus \Delta_{ni} \cup D_{ni}),$$

therefore there exists an isotopy $\Psi^{n+1} : E^3 \rightarrow E^3$ such that

$$\Psi_0^{n+1} = 1; \quad \Psi_t^{n+1} \Big|_{E^3 \setminus \bigcup_{i=1}^{k_n} u'_{ni}} = 1; \quad \Psi_t^{n+1} \Big|_{S_{n+1}} = \psi^{n+1}.$$

Put

$$H_t^{n+1} = \Gamma_t^{n+1} \circ (\Psi_t^{n+1} \Gamma_1^{n+1}); \quad \tilde{H}_t^{n+1} = H_t^{n+1} \Phi_1^n.$$

From (2), by induction we find

$$\Phi_1^{n+1} \Big|_{\sigma_2} = g_{n+1} f. \tag{7}$$

3. The sequence of homeomorphisms $\Phi_1^n : E^3 \rightarrow E^3$ converges uniformly to a continuous mapping $F : E^3 \rightarrow E^3$. Since Φ_1^n is joined to Φ_1^{n+1} by the ε_n -isotopy \widetilde{H}_t^{n+1} , then, after making for each isotopy \widetilde{H}_t^n a linear change of the variable t , $\chi_n : [0, 1] \rightarrow [1 - 1/2^{n-1}, 1 - 1/2^n]$, and putting

$$F_0 = H_0^1 = 1; \quad F_t = \widetilde{H}_{\chi_n^{-1}(t)}, \quad 1 - 1/2^{n-1} \leq t \leq 1 - 1/2^n; \quad F_1 \equiv F = \lim_{n \rightarrow \infty} \Phi_1^n,$$

we obtain a pseudo-isotopy $F_t : E^3 \rightarrow E^3$. The relation $F_1|_{\sigma^2} \equiv f$ follows from (7).

It is easily proved that the set K of points whose inverse images are degenerate is, for any n , contained in

$$V_n = \bigcup_{k=n}^{\infty} \bigcup_{i=1}^{n_k} u_{ki}.$$

By (4), V_n is the sum of pairwise nonintersecting regions of diameter $< \varepsilon_n$, and $u_{ki} \cap S^2 \neq \Lambda$. Therefore $K \subset S^2$; $\dim K = 0$. Since $u_{ki} \cap S^2$ is a wild disk, K is contained in the set of wildness points of S^2 , and the conditions of Theorem 1 are satisfied.

Let us note that, for an embedding in E^3 of the one-dimensional sphere S^1 , a theorem analogous to Theorem 1 is impossible, since tame spheres Σ^1 approximating the wild sphere S^1 may, when approaching S^1 , form increasingly complicated knots.

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
19 XI 1965

REFERENCES

1. J. W. Alexander, Proc. Nat. Acad. Sci. U.S.A., **10**, 1 (1924).
2. R. H. Bing, Ann. Math., **59**, 145 (1954).
3. R. H. Bing, Ann. Math., **65**, No. 3, 456 (1957).
4. R. H. Bing, Ann. Math. J., **84**, 583 (1962).
5. R. H. Bing, Michig. Math. J., **11**, No. 1, 33 (1964).
6. H. Gluck, Bull. Am. Math. Soc., **69**, No. 6, 824 (1963).
7. M. Kister, Trans. Am. Math. Soc., **97**, No. 2, 213 (1960).

8. E. E. Moise, *Ann. Math.*, **59**, 159 (1954).
9. D. E. Sanderson, *Am. Math. Soc. Proc.*, **11**, 171 (1960).
10. A. V. Chernavskii, *DAN*, **165**, No. 6 (1965).
11. L. L. Linninger, *Trans. Am. Math. Soc.*, **118**, No. 6, 534 (1965).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.