ON EXTENSIONS AND REMAINDERS OF FINITE ORDER OF COMPLETELY REGULAR SPACES

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ON EXTENSIONS AND REMAINDERS OF FINITE ORDER OF COMPLETELY REGULAR SPACES

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In this note all topological spaces considered are completely regular Hausdorff spaces, and all mappings are continuous.

A sequence

$$(C^0(X),\ b_1C^0(X),\ b_2C^1(X),\ \dots,\ b_{n-1}C^{n-2}(X),\ b_{nC}^{n-1}(X))$$

of spaces, where $C^0(X) = X$; $b_{iC}^{i-1}(X)$ is some compact extension of $C^{i-1}(X)$ and $C^i(X) = b_{iC}^{i-1}(X) \setminus C^{i-1}(X)$ for $1 \le i \le n$, is called a **compact extension of n-th order** of the space X, and $C^n(X)$ an **n-th order remainder** of the space X. In the case where all b_i are Čech extensions β_i, the extension is called the **Čech extension of n-th order** of the space X, and the remainder of n-th order $\Gamma^n(X)$ the **Čech remainder of n-th order** of the space X.

Let M_n^* be the class of all compact extensions of n-th order of the space X. We introduce in M_n^* a quasi-ordering as follows:

$$(\widetilde{C}^0(X),\ \tilde{b}_1\widetilde{C}^0(X),\ \tilde{b}_2\widetilde{C}^1(X), \dots, \tilde{b}_n\widetilde{C}^{n-1}(X)) \ge (C^0(X),\ b_1C^0(X),\ b_2C^1(X), \dots,\ b_{nC}^{n-1}(X)),$$

if there exist mappings

$$f_i : \tilde{b}_i\widetilde{C}^{i-1}(X) \to b_{iC}^{i-1}(X), \quad i = 1, \dots, n,$$

such that f_1 is an extension of the identity mapping $e : \widetilde{C}^0(X) \to C^0(X)$, and for $i = 2, \dots, n$ the mapping f_i is an extension, on $\tilde{b}_i\widetilde{C}^{i-1}(X)$, of the mapping induced by f_{i-1}. If all f_i are homeomorphisms, then we shall call such extensions **equivalent**.

Let M_n be the set of classes of equivalent extensions. Using Lemma 1.5 of the paper (^{1}), one can prove the following lemma.

Lemma 1. *The set M_n is a partially ordered set whose largest element is the class of the Čech extension of n-th order.*

It follows from Lemma 1 that if some n-th order remainder of the space X has a perfect property (see (^{1})), then all its n-th order remainders have this property.

Denote by P_n the following property of a space: the n-th order remainder of the space has the perfect property P. Then from Lemma 1.5 (^{1}) and from Lemma 1 the following proposition follows immediately.

Lemma 2. *The property P_n is a perfect property.*

Denote by $R^1(X)$ the set of all points of the space X that have no compact neighborhoods, and let, for $n > 1$,

$$R^n(X) = R^1[R^{n-1}(X)]$$

(see (^{1})). We shall assume that $X = R^0(X)$.

Let K_n, where $n \geq 0$, denote the class of all spaces whose n-th order remainder is compact. By a space with compact remainder of zeroth order we mean a compact space. It can be shown that there exists a space belonging to K_n and not belonging to K_{n-1}, and that if the space X has compact

if it is a remainder of finite order, then it contains an open everywhere dense locally compact subset A, and $X \setminus A = R^1(X)$.

Since compactness is an absolute property, Lemma 2 implies the following theorem.

Theorem 1. *If f is a perfect mapping of X onto Y, then X belongs to the class K_n if and only if Y belongs to the class K_n.*

Theorem 2. *A space X belongs to the class K_n if and only if $R^{n/2}(X)$ is compact, when n is even, and $R^{(n-1)/2}(X)$ is locally compact, when n is odd.*

This theorem follows from the fact that $R^j(X)$ is a remainder of order $2j$ of the space X for $j \geqslant 1$.

From Theorem 2, in particular, Theorem 3.1 of Henriksen and Isbell (^{1}) follows: X is locally compact at infinity (i.e., has a compact remainder of second order) if and only if $R^1(X)$ is compact. Further, X has a locally compact remainder of second order if and only if $R^1(X)$ is locally compact.

It is easy to see that if F is a closed subset of the space X, then $R^k(F)$ is a closed subset of $R^k(X)$. Hence it follows that if $X \in K_n$, then $F \in K_n$. Therefore, if X is a compact space and A is its subset belonging to K_n, then $X \setminus A$ belongs to K_{n+1}.

Theorem 3. *Let A and B be subsets of a space X, with A compact and B a space in K_n. Then, for even n, $A \cup B \in K_n$, and for odd n, $A \cup B \in K_{n+1}$.*

Proof. For $n = 0$ this is known. Let bX be a compact extension of X. We have

$$\overline{A \cup B} \setminus A \cup B = (\overline{B} \setminus B) \setminus A.$$

The closure is taken in X. Let $\overline{B} \setminus B = D$. It is clear that

$$[A \cup (\overline{D} \setminus D)] \cap (D \setminus A) = \emptyset \quad \text{and} \quad [A \cup (\overline{D} \setminus D)] \cup (D \setminus A) = \overline{D} \cup A.$$

Let us prove the theorem for $n = 1, 2$. If $B \in K_1$, then $D \in K_0$ and $\overline{D} \setminus D = \emptyset$. Then $D \setminus A \in K_1$, and therefore $A \cup B \in K_2$. If $B \in K_2$, then $D \in K_1$ and $\overline{D} \setminus D \in K_0$. Since $A \cup (\overline{D} \setminus D) \in K_0$, we obtain that $D \setminus A \in K_1$, and therefore $A \cup B \in K_2$. Suppose the theorem is proved for $n = m, m+1$; we shall prove it for $n = m+2, m+3$. If $B \in K_{m+2}$, then $\overline{D} \setminus D \in K_m$. By induction $A \cup (\overline{D} \setminus D) \in K_{m+1}$. Then $D \setminus A \in K_{m+2}$, and therefore $A \cup B \in K_{m+3}$. If $B \in K_{m+3}$, then $\overline{D} \setminus D \in K_{m+1}$. By induction $A \cup (\overline{D} \setminus D) \in K_{m+1}$. Therefore we obtain that $D \setminus A \in K_{m+2}$ and, hence, $A \cup B \in K_{m+3}$. The theorem is proved.

Using Theorem 3, one can prove the following propositions:

1. The direct product of a compact space and a space in K_n is a space in K_n.
2. The direct product of a locally compact space and a space in K_n is a space in K_{n+1} for even n, and a space in K_n for odd n.
3. If X is a space in K_n and A is its compact subset, then $X \setminus A$ is a space in K_{n+1} for even n, and a space in K_n for odd n.
4. If X is a locally compact space and A is its subset belonging to the class K_n, then $X \setminus A$ is a space in K_{n+1} for even n, and a space in K_{n+2} for odd n.

From 4, in particular, it follows that the complement of a locally compact set in a locally compact space is a space for which the set of points at which the space is not locally compact is locally compact.

Let f be a mapping of X onto Y. Denote by $R(f)$ the set of all points $y \in Y$ such that $f^{-1}(y)$ is noncompact. By a theorem of Stone ([2]), Theorem 88, f has a unique extension $f^* : \beta X \to \beta Y$.

Theorem 4. *If f is a closed mapping of X onto Y, then*

$$f^*[\Gamma^1(X)] \setminus \Gamma^1(Y) = R(f).$$

Proof. It is clear that $R(f) \subset \bar{f}^*[\Gamma^1(X)] \setminus \Gamma^1(Y)$. Let $b \in \Gamma^1(X)$ and $f^*(b) = a' \in f^*[\Gamma^1(X)] \setminus \Gamma^1(Y)$. Suppose that $f^{-1}(a') = A$ is compact. Since A is closed in βX and $b \notin A$, there exists a continuous function $\varphi : \beta X \to I = [0,1]$ such that $\varphi(a) = 0$ for $a \in A$ and $\varphi(b) = 1$. Let

$$Q = \{x \in X,\ \varphi(x) \geq 1/2\}.$$

It is clear that Q is closed in X, and therefore $f(Q) = Q'$ is closed in Y, i.e. $Q' = P \cap Y$, where P is closed in βY. Obviously, $a' \notin Q'$. Then there exists a continuous function $\psi : \beta Y \to I$ such that $\psi(a') = 0$ and $\psi(p) = 1$ for $p \in P$. Let $k = \psi f^*$. Then $k(b) = 0$. Denote by D the set of points x of the space X for which $\varphi(x) > 1/2$ and $k(x) < 1/2$. It is clear that $b \in D$. The set D is open, and therefore there exists a point $s \in X$ such that $s \in D$, i.e. $k(s) < 1/2$ and $\varphi(s) > 1/2$. Further, we have $\psi[f^*(s)] < 1/2$, and therefore $f^*(s) = f(s) \notin Q'$. Hence it follows that $s \notin Q$, and therefore $\varphi(s) < 1/2$. The contradiction obtained shows that $f^{-1}(a')$ is noncompact, i.e. $a' \in R(f)$. The theorem is proved.

Using Theorems 3 and 4, one can prove the following theorem.

Theorem 5. *Let f be a closed mapping of X onto Y such that the $R(f)$-space is of class K_n. Then, if $X \in K_1$, then $Y \in K_{n+2}$, and if $X \in K_2$, then for even n, $Y \in K_{n+2}$, while for odd n, $Y \in K_{n+3}$.*

In particular, it follows from Theorem 5 that under a closed mapping of a locally compact space, if the set of points whose full inverse image is noncompact is compact, then the image is a space for which the set of points having no compact neighborhoods is compact.

Let f be a mapping of X onto Y, and let $f^* : bX \to bY$ be an extension of f to their compact extensions bX and bY. Let $X_1 = bX \setminus X$, $Y_1 = f^*(X_1)$, and let f_1 be the mapping of X_1 onto Y_1 induced by the mapping f^*. Then the mapping $f_1 : X_1 \to Y_1$ will be called a **remainder of first order** (or simply a **remainder**) of the mapping f. If bX and bY are Čech compactifications, then f_1 is called the **Čech remainder** of the mapping f. The mapping $f_n : X_n \to Y_n$ is called a **remainder of n-th order of the mapping** f, if f_n is a remainder of first order of a remainder of $(n-1)$-st order of the mapping f. It is easy to see that if some remainder of a closed mapping is a perfect mapping, then all remainders are also perfect. Such a mapping f will be called **perfect at infinity**.

With the aid of Lemma 1.5 ([1]) and Theorem 4 one can prove the following theorem.

Theorem 6. *A closed mapping of X onto Y is perfect at infinity if and only if, for every point $x \in X$ having no compact neighborhoods, the full inverse image of its image is compact.*

Theorem 7. *A closed mapping of X onto Y with a closed Čech remainder is perfect at infinity if and only if the full inverse image of every point of Y is locally compact.*

With the help of Theorems 1, 3, and 4 we obtain the following theorem.

Theorem 8. *Let f be a closed mapping of X onto Y, perfect at infinity, such that $R(f)$ is compact. Then: 1) if $X \in K_n$, then $Y \in K_n$ for even n and $Y \in K_{n+1}$ for odd n; 2) if $Y \in K_n$, then $X \in K_{n+1}$ for even n and $X \in K_n$ for odd n.*

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2. M. H. Stone, Trans. Am. Math. Soc., **41**, 375 (1937).

Note: Figure translations are in progress. See original paper for figures.