MOTION OF A PISTON IN A HEAT-CONDUCTING AND VISCOUS MEDIUM

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MOTION OF A PISTON IN A HEAT-CONDUCTING AND VISCOUS MEDIUM

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Consider the one-dimensional motion of a gas ahead of a piston that moves with constant velocity U in a heat-conducting and viscous medium. The basic equations of one-dimensional unsteady motion of a viscous and heat-conducting medium have the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial r}+(\nu-1)\frac{\rho u}{r}=0,$$

$$\frac{\partial\left(r^{\nu-1}\rho u\right)}{\partial r}+\frac{\partial}{\partial r}\left[r^{\nu-1}\left(\rho u^{2}-p_{rr}\right)\right]+(\nu-1)r^{\nu-2}p_{nn}=0,$$

$$\frac{\partial}{\partial t}\left[\rho r^{\nu-1}\left(\frac{u^{2}}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial r}r^{\nu-1}\left[\rho u\left(\frac{u^{2}}{2}+\varepsilon\right)-p_{rr}u-\chi\frac{\partial T}{\partial r}\right]=0, \quad (1)$$

$$p_{rr}=-p+\lambda\left[\frac{\partial u}{\partial r}+(\nu-1)\frac{u}{r}\right]+2\mu\frac{\partial u}{\partial r},$$

$$p_{nn}=-p+\lambda\left[\frac{\partial u}{\partial r}+(\nu-1)\frac{u}{r}\right]+2\mu\frac{u}{r},$$

where t is time; r is the coordinate; ρ is density; u is the velocity component; ε is internal energy; T is temperature; χ is the coefficient of heat conductivity; p is pressure; p_{rr}, p_{nn} are stress coordinates; λ, μ are viscosity coefficients; $\nu = 1, 2, 3$, respectively, for plane, cylindrical, and spherical pistons. Suppose that $p = R\rho T$ and $\varepsilon = C_{vT}$, where R is the gas constant and C_v is the specific heat conductivity.

Instead of the dimensional independent variables t and r, take the dimensionless independent variables

$$\xi=\frac{\gamma p_{1}}{\rho_{1}}\frac{t^{2}}{r^{2}} \quad \text{and} \quad \eta=\frac{\chi t}{\rho_{1}C_{v}r^{2}},$$

where ρ_1, p_1 are the density and pressure ahead of the shock wave. We also introduce the dimensionless constants $q_1 = 2\mu C_v/\chi$ and $q_2 = \lambda C_v/\chi$. The sought dimensional functions are represented in terms of dimensionless functions depending on the dimensionless variables by

$$u = \frac{r}{t} V_1(\xi, \eta), \qquad p = \rho_1 \frac{r^2}{t^2} P_1(\xi, \eta), \qquad \rho = \rho_1 R_1(\xi, \eta). \tag{2}$$

Let $V_0(\xi), P_0(\xi), R_0(\xi)$ be the solutions of the problem for self-similar motion obtained by L. I. Sedov ([1]). Then the required linearized solutions can be represented in the form

$$V_1 = V_0(\xi) + \eta V(\xi), \qquad P_1 = P_0(\xi) + \eta P(\xi), \qquad R_1 = R_0(\xi) + \eta R(\xi). \tag{3}$$

After passing to dimensionless variables and varying with respect to η, we obtain a system of ordinary differential equations for V, P, R. Assuming that the velocity of the piston is large and that ξ^2 may be neglected, the system takes the form

$$2\xi(1-V_0)\, dR/d\xi - 2\xi R_0\, dV/d\xi + [(\nu-2)R_0 - 2\xi\, dR_0/d\xi]\, V + [(\nu-2)V_0 + 1 - 2\xi\, dV_0/d\xi]\, R = 0,$$

$$\begin{aligned} &2\xi(1-2V_0)R_0\, dV/d\xi + 2\xi V_0(1-V_0)\, dR/d\xi - 2\xi\, dP/d\xi \\ &\quad + 2\left\{\xi(1-2V_0)\, dR_0/d\xi + [(\nu-1)V_0 - 2\xi\, dV_0/d\xi]\, R_0\right\} V \\ &\quad + \left[2\xi(1-2V_0)\, dV_0/d\xi + \nu V_0^2\right] R + 2\xi(\nu-2)(q_1+2q_2)\, dV_0/d\xi = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} &\left[V_0R_0(2-3V_0) - \frac{2\gamma P_0}{\gamma-1}\right]\xi\frac{dV}{d\xi} + 2\xi\frac{1-\gamma V_0}{\gamma-1}\frac{dP}{d\xi} + \xi V_0^2(1-V_0)\frac{dR}{d\xi} + \\ &+ \left[2\xi V_0\left(1-\frac{3}{2}V_0\right)\frac{dR_0}{d\xi} + 2\xi R_0(1-3V_0)\frac{dV_0}{d\xi} - V_0R_0 + \frac{3}{2}\nu V_0^2 R_0 + \right. \\ &\qquad \left. + \frac{\nu\gamma}{\gamma-1}P_0 - \frac{2\xi\gamma}{\gamma-1}\frac{dP_0}{d\xi}\right] V + \frac{1}{\gamma-1}\left(\nu\gamma V_0 - 1 - 2\xi\gamma\frac{dV_0}{d\xi}\right) P + \\ &+ \left[\frac{V_0^2}{2}(\nu V_0 - 1) + 2\xi V_0\left(1-\frac{3}{2}V_0\right)\frac{dV_0}{d\xi}\right] R - \nu(q_1+\nu q_2)V_0^2 + \\ &+ (\nu q_1 + (3\nu-2)q_2)\left[2\xi V_0\frac{dV_0}{d\xi} - \frac{2\gamma}{\gamma-1}\left(\frac{P_0}{R_0} - \frac{\xi}{R_0}\frac{dP_0}{d\xi} + \xi\frac{P_0}{R_0^2}\frac{dR_0}{d\xi}\right)\right] = 0. \end{aligned}$$

First we solve the system of homogeneous differential equations.

Fig. 1

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

Fig. 2

We denote these solutions by $\overline{V}, \overline{P}, \overline{R}$. From the independent variable ξ we pass to the independent variable V_0 and seek solutions in the form

$$\overline{V} = (1 - V_0)^s \sum_{n=0}^{\infty} a_{ni}(1 - V_0)^n, \qquad \overline{P} = (1 - V_0)^s \sum_{n=0}^{\infty} b_{ni}(1 - V_0)^n,$$

$$\overline{R} = (1 - V_0)^s \sum_{n=0}^{\infty} c_{ni}(1 - V_0)^n, \qquad i = 1, 2, 3, 4. \tag{5}$$

We also represent ξ, P_0, R_0 as series in powers of $(1 - V_0)$

$$\xi = \xi_n \left[1 - \frac{2}{\nu}(1 - V_0) + \cdots\right], \qquad P_0 = P_{0n}\left[1 - \frac{2}{\nu}(1 - V_0) + \cdots\right],$$

$$R_0 = R_{0n}\left[1 - \frac{\nu - 1}{2\nu}\frac{R_{0n}}{\gamma P_{0n}}(1 - V_0)^2 + \cdots\right]; \tag{6}$$

ξ_n, P_{0n}, R_{0n} are the values of the functions at the piston.

The characteristic equation of the system is

$$s^2\left[\frac{\gamma}{\gamma - 1}\frac{P_{0n}}{R_{0n}}\left(s + \frac{1}{\nu}\right) - \frac{1}{\nu}\right] = 0. \tag{7}$$

The roots $s_1 = s_2 = 0$ correspond to solutions with a logarithmic singularity ([2]). For $\gamma = 1.4$, the characteristic root $s_3 = 13/35,\ 5/14,\ 29/105$, respectively for $\nu = 1, 2, 3$.

Particular solutions of the system of differential equations with the right-hand side are likewise sought in the form of series in powers of $(1 - V_0)$.

The required solutions must satisfy the boundary conditions at the piston and at the shock wave. At the piston $u = U$, and since $V_0 = 1$, then

$V(1) = 0$. Ahead of the shock wave the gas is at rest, the density and pressure are constant, and therefore the conditions at the shock wave can be written in the form

$$\rho_1 c = \rho_2(c - u_2), \qquad p_{rr_1} = \rho_1 c u_2 + p_{rr_2}, \tag{8}$$

Fig. 3

Figure 2: Fig. 3

$$\frac{cp_1}{\gamma-1}=\rho_1 c\left(\frac{u_2^2}{2}+\frac{p_2}{(\gamma-1)\rho_2}\right)+p_{rr_2}u_2+\frac{\kappa}{C_v(\gamma-1)}\frac{\partial}{\partial r}\frac{p_2}{\rho_2},$$

where c is the velocity of the shock wave. The conditions at the shock wave after passing to dimensionless variables and varying with respect to η have the form

$$\left(R_2\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*}-2\xi^* a\left.\frac{dR_0}{dV_0}\right|_{V_0=V_0^*}\right)(1-V_0^*)-$$

$$-R_0\left[(aV_0^*+V_2)\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*}+2a\xi^*\right]=0$$

$$(aP_0^*+P_2)\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*}-2a\xi^*\left.\frac{dP_0}{dV_0}\right|_{V_0=V_0^*}-(\nu q_2+q_1)V_0^*\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*}+$$

$$+(a+q_1+q_2)2\xi^*=V_2\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*}, \tag{9}$$

$$2\frac{1}{\gamma-1}\left(\frac{P_0^*}{R_0}\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*}-\xi^*\left.\frac{d}{dV_0}\frac{P_0}{R_0}\right|_{V_0=V_0^*}\right)+aV_0^*\left(V_0^*|_{V_0=V_0^*}+2\xi^*\right)=$$

$$=\left(V_0^*V_2-\frac{P_2}{R_0}+\frac{P_0^*}{R_0^2}R_2\right)\left.\frac{d\xi}{dV_0}\right|_{V_0=V_0^*},$$

where V_0^* is the value of V_0 at the shock wave, and a is a constant.

The radius vector of the shock wave is represented in the form

$$r_2=r_{20}(1+a\eta+...), \tag{10}$$

where r_{20} is the radius vector of the shock wave for self-similar motion.

The solutions for V, P, and R have the following form, if one retains only $1-V_0$ to the first degree:

Fig. 3

$$V = (1 - V_0)\left\{[1 + \ln(1 - V_0)]a_{11} + \left.\frac{\partial a_{11}}{\partial s}\right|_{s=0} + a_1\right\} + \ldots,$$

$$P = [1 + \ln(1 - V_0)][b_{01} + b_{11}(1 - V_0)] + \left(\left.\frac{\partial b_{11}}{\partial s}\right|_{s=0} + b_1\right)(1 - V_0) + \ldots,$$

$$R = [1 + \ln(1 - V_0)][c_{01} + c_{11}(1 - V_0)] + \left.\frac{\partial c_{01}}{\partial s}\right|_{s=0} + \left.\frac{\partial c_{11}}{\partial s}\right|_{s=0}(1 - V_0) +$$

$$+ c_{03}(1 - V_0)^{s_3} + c_0 + c_1(1 - V_0) + \ldots \quad (11)$$

In these formulas the condition at the piston has already been taken into account. The constants b_{01}, c_{03}, and a are found from the conditions at the shock wave, while the constants a_1, b_1, c_0, c_1 are known coefficients of particular solutions of the inhomogeneous system.

Computations of V, P, and R were carried out for Prandtl number $\mathrm{Pr} = 1$ for $\nu = 1, 2, 3$; they are presented in the form of graphs (see Figs. 1, 2, and 3).

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References

1. L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Moscow, 1961.

2. G. T. Padé, *Integration of Differential Equations*, Moscow–Leningrad, 1933.

Note: Figure translations are in progress. See original paper for figures.