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MATHEMATICS

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ON FACTOR REPRESENTATIONS OF ANTI-COMMUTATION RELATIONS

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1. A **representation of the anticommutation relations** is a set of bounded linear operators $\{a_k\}_1^\infty$, acting in a separable Hilbert space H, which satisfy the system of equalities

$$a_j a_k + a_k a_j = 0, \qquad a_j a_k^* + a_k^* a_j = \delta_{jk}. \tag{1}$$

If the weakly closed ring generated by the operators $\{a_k, a_k^*\}_1^\infty$ is a factor, then such a representation will be called a **factor representation**.

Factor representations of the relations (1) were first studied by von Neumann; he constructed examples of factors of types I, II, III, which are generated by the operators $\{a_k, a_k^*\}_1^\infty$ ([2]). In the present paper several general results are obtained on the construction of factor representations and on their properties. We note that the simplest properties are possessed by those factor representations for which the weakly closed ring generated by the operators $\{a_k, a_k^*\}_1^\infty$ is a factor Π_1.

2. Our considerations are based on the work ([1]). In that work a general construction was given for representations of the relations (1), which was used by the authors to study irreducible representations. We shall apply this construction to the study of factor representations. Let us recall some results of ([1]).

By virtue of (1), the operators $N_k = a_k^* a_k$ form a commuting set of projections. We diagonalize these projections. It is convenient to proceed as follows. Let Γ be the set of all infinite sequences $\alpha = (\alpha_1, \alpha_2, ...)$, where $\alpha_i = 0, 1$. Put $\Gamma_k = (\alpha; \alpha_k = 0)$. By a Borel set in Γ we shall mean a set constructed from all Γ_k by the usual countable processes. We now represent the space H in the form of an integral $\int \oplus H_\alpha d\mu(\alpha)$ over Γ in such a way that if to an element $f \in H$ there corresponds the vector-function $f(\alpha)$ with values in the Hilbert space H_α, then $N_k f(\alpha) = \alpha_k f(\alpha)$; $\mu(\alpha)$ is a completely additive bounded nonnegative function of Borel sets in Γ.

Under componentwise addition modulo 2, Γ is a group. Let δ_k be the sequence from Γ with a one in the k-th place and zeros elsewhere. From the relations (1) it follows that the measures $\mu(\alpha)$ and $\mu(\alpha+\delta_k)$ are equivalent. Denote $\lim H_\alpha = \nu(\alpha)$. For factor representations, $\nu(\alpha)$ is always constant.

Introduce the operators

$$A_k = a_k + a_k^*, \qquad B_k = i^{-1}(a_k - a_k^*).$$

In ([1]) the formulas

$$A_k f(\alpha) = j_k(\alpha) c_k(\alpha) \sqrt{\frac{d\mu(\alpha+\delta_k)}{d\mu(\alpha)}}\, f(\alpha+\delta_k), \tag{2}$$

$$B_k f(\alpha) = i^{-1} j_{k+1}(\alpha) c_k(\alpha) \sqrt{\frac{d\mu(\alpha+\delta_k)}{d\mu(\alpha)}}\, f(\alpha+\delta_k),$$

were found.

where $j_k(a) = (-1)^{a_i+\cdots+a_{k-1}}$, and $\{c_k(a)\}_1^\infty$ is a measurable unitary transformation from H_a into $H_{a+\delta_k}$, satisfying the relations

$$c_k(a+\delta_k) = c_k^*(a); \qquad c_k(a)c_l(a+\delta_k) = c_l(a)c_k(a+\delta_l) \tag{3}$$

for almost all a and all k, l.

Putting $T_k a = (0, \ldots, 0, a_{k+1}, a_{k+2}, \ldots)$, let us introduce the notation

$$\gamma_k(a) = c_k(T_k a) \tag{4}$$

and, using (3), we obtain

$$c_k(a) = \gamma_1^{-a_1}(a) \cdots \gamma_{k-1}^{-a_{k-1}}(a) \gamma_k^{(-1)^{a_k}}(a) \gamma_{k-1}^{a_{k-1}}(a+\delta_k) \cdots \gamma_1^{a_1}(a+\delta_k). \tag{5}$$

Conversely, if $\{\gamma_k(a)\}_1^\infty$ are arbitrary unitary measurable transformations from H_a into H_a that are invariant under shifts by $\delta_1, \ldots, \delta_k$, then the $c_k(a)$ defined by formulas (5) satisfy the functional equations (3).

In (1) it was shown that a representation of the anticommutation relations is determined by specifying a measure μ, a dimension function ν, and a set of operators $c_k(a)$ satisfying (3). Therefore it is sometimes convenient to denote a representation by $(\mu, \nu, \{c_k(a)\}_1^\infty)$.

A quasi-invariant measure is called **ergodic** if every bounded measurable function $f(a)$ satisfying the condition $f(a) = f(a + \delta_k)$ for all k and almost all a is constant.

The simplest examples of quasi-invariant ergodic measures can be constructed as follows. We shall regard the set Γ as the direct product of a countable set of copies of two-point sets. Put $\mu(\Gamma) = 1$, $\mu(\Gamma_k) = p_k$, $0 < p_k < 1$, and define the measure μ on Γ as the product measure. If $p_k = 1/2$ for all k, then such a measure is called **Lebesgue**.

3. We proceed to the study of factor representations.

Theorem 1. *If a representation is a factor representation, then the measure $\mu(a)$ is ergodic.*

Theorem 2. *A factor representation of the anticommutation relations for which $\nu < \infty$ is a direct sum of q copies of some irreducible representation, where q divides ν.*

4. Let H be the Hilbert space of vector-functions $f(x, y)$ on $\Gamma \times \Gamma$, whose values belong to a Hilbert space R, and

$$\int_{\Gamma\times\Gamma} \|f(x, y)\|^2 \, d\mu_1(x) \, d\mu_2(y) < \infty,$$

where $\|\cdot\|$ is the norm of a vector in R, and $\mu_1(x)$, $\mu_2(y)$ are quasi-invariant ergodic measures.

We define representations of the anticommutation relations by the formulas

$$A_k f(x, y) = j_k(x) c_k(x, y) f(x + \delta_k, y) \sqrt{\frac{d\mu_1(x + \delta_k)}{d\mu_1(x)}}, \tag{6}$$

$$B_k f(x, y) = i^{-1} j_{k+1}(x) c_k(x, y) f(x + \delta_k, y) \sqrt{\frac{d\mu_1(x + \delta_k)}{d\mu_1(x)}}.$$

Define the dual representation

$$\widetilde{A}_k f(x, y) = j_k(y) \, \tilde{c}_k(x, y) \, f(x, y + \delta_k) \sqrt{\frac{d\mu_2(y + \delta_k)}{d\mu_2(y)}},$$

$$\widetilde{B}_k f(x, y) = i^{-1} j_{k+1}(y) \, \tilde{c}_k(x, y) \, f(x, y + \delta_k) \sqrt{\frac{d\mu_2(y + \delta_k)}{d\mu_2(y)}}. \tag{7}$$

Here $j_k(x), j_k(y)$ are defined by formula (2), and $c_k(x, y)$ ($\tilde{c}_k(x, y)$) is a measurable unitary transformation in the space R, satisfying relations (3) for fixed y

(respectively, for fixed x). Suppose that each operator A_k, B_k commutes with each operator $\widetilde{A}_l, \widetilde{B}_l$. For this it is necessary and sufficient that the relations

$$c_k(x,y)\tilde{c}_l(x+\delta_k, y) = \tilde{c}_l(x,y)c_k(x, y+\delta_l) \tag{8}$$

hold for all k, l and almost all $(x, y) \in \Gamma \times \Gamma$.

The following theorem shows that the construction just described is of a general character.

Theorem 3. Let $\{A_k, B_k\}_1^\infty$ and $\{\widetilde{A}_k, \widetilde{B}_k\}_1^\infty$ be factor representations of relations (1), and let the operators of both representations act in the same Hilbert space H. If each operator A_k, B_k commutes with each operator $\widetilde{A}_l, \widetilde{B}_l$, then the space H can be realized in such a way that the operators A_k, B_k $(k = 1, 2, ...)$ have the form (6), while the operators $\widetilde{A}_k, \widetilde{B}_k$ $(k = 1, 2, ...)$ have the form (7).

Let us return to our construction. Denote by M the weakly closed ring generated by the operators (6), and by $\widetilde{M}$ the weakly closed ring generated by the operators (7).

Theorem 4. If, for almost every fixed y, relations (6) define an irreducible representation, and, for almost every fixed x, relations (7) also define an irreducible representation and, moreover, the operators A_k, B_k $(k = 1, 2, ...)$ commute pairwise with the operators $\widetilde{A}_l, \widetilde{B}_l$ $(l = 1, 2, ...)$, then M and $\widetilde{M}$ are factors, and the weakly closed ring generated by the operators from $M \cup \widetilde{M}$ is the ring of all bounded linear operators in the space H.

If the conditions of Theorem 4 are fulfilled, then the functions $c_k(x, y)$ and $\tilde{c}_k(x, y)$ satisfy two systems of functional equations (3) and (8). We indicate some solutions of these equations.

The simplest solution is obtained if one assumes that $c_k(x, y) = c_k(x)$ and $\tilde{c}_k(x, y) = \tilde{c}_k(y)$. Then (8) reduces to the requirement that $c_k(x)$ and $\tilde{c}_l(y)$, for all k, l and almost all x and y, commute with one another.

Consider another solution. Suppose that $\tilde{c}_k(x, y) = c_k(x, y)$. Then it follows from (8) that, for almost all x and y and all k,

$$c_k(x+\delta_k, y) = c_k(x, y+\delta_k). \tag{9}$$

If the measures μ_1 and μ_2 are product measures, then it follows from (9) that $c_k(x, y) = c_k(x + y)$. Conversely, if this condition is fulfilled, then from it and from (3) condition (8) follows.

Theorem 5. For every irreducible representation $(\mu, \nu, \{c_k(a)\}_1^\infty)$ one can construct a factor representation which will have the form (6), with $c_k(x, y) = \tilde{c}_k(x, y) = c_k(x + y)$, $(k = 1, 2, ...)$.

The theorem makes it possible to construct a large class of factor representations. For example, to every irreducible representation with $\nu < \infty$, constructed in (1), there corresponds a nontrivial factor representation. It is relatively easy to determine whether it is of type I or not. However, the exact type can be established only in the simplest cases, which we shall now consider.

Let in (6) and (7) $c_k(x, y) = \tilde{c}_k(x, y) = i(-1)^{x_k + y_k}$; μ_1, μ_2 are Lebesgue measures; $f(x, y)$ are complex-valued functions with integrable square of the modulus. Then M and $\widetilde{M}$ are factors of type II_1. If one assumes that μ_2 is not equivalent to Lebesgue measure, then $\widetilde{M}$ will have type III.

5. In this section we shall consider factor representations of the relations (1) in connection with factors of type II_1.

Theorem 6. If the operators $\{A_k, B_k\}_1^\infty$ generate a factor of type II_1, then

$$\mathrm{Tr}\left(B_{i_1}^{\alpha_1} A_{i_1}^{\beta_1}, \dots, B_{i_n}^{\alpha_n} A_{i_n}^{\beta_n}\right) = 0,$$

where Tr is the relative trace in the factor of type II_1, $\alpha_i, \beta_i = 0, 1$.

Theorem 7. If the condition of Theorem 6 is satisfied, then the measure $\mu(\alpha)$ is equivalent to Lebesgue measure.

Theorem 8. There exist representations of the anticommutation relations whose operators generate spatially nonisomorphic factors of type II_1.

6. We give one more method for constructing representations.

Theorem 9. Let a free commutative group with a finite number of generators $a_1, \dots, a_t$ be given. Let U be a unitary projective representation of the group G. Construct a representation of the anticommutation relations for which: 1) $\gamma_{nt+i} = U(a_i)$, where $0 < i \leq t$, $n = 1, 2, \dots$; 2) $\mu(\alpha)$ is a product of measures, with $\mu(\Gamma_k) = p_k$ having neither 0 nor 1 as limit points.

Then, for this representation of the anticommutation relations, the commutant is algebraically isomorphic to the commutant of the representation U of the group G.

With the aid of this theorem one can construct irreducible representations of the relations (1) for $\nu = \infty$ ([1,4]), as well as representations for which the ring generated by $\{A_k, B_k\}_1^\infty$ is a factor of type II_∞, and the measure $\mu(\alpha)$ is not equivalent to Lebesgue measure.

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Note: Figure translations are in progress. See original paper for figures.