ON THE QUESTION OF THE MOTION OF CHARGED PARTICLES IN AN ACUTE-ANGLE TRAP

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Abstract

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ON THE QUESTION OF THE MOTION OF CHARGED PARTICLES IN AN ACUTE-ANGLE TRAP

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In works (^{1}, ^{2}), regions of possible motion of particles in magnetic traps with opposing fields were studied. The magnetic field in such a trap, in the simplest case, has the form

$$H_\rho = -A\rho; \qquad H_\varphi = 0; \qquad H_z = 2Az. \tag{1}$$

The equations of motion of a particle with mass m and charge e in the field (1) have the form

$$\ddot{\rho} - \rho\dot{\varphi}^2 = 2A_0\rho z\dot{\varphi}, \tag{2a}$$

$$2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} = -A_0(\rho\dot{z} + 2z\dot{\rho}), \tag{2b}$$

$$\ddot{z} = A_0\rho^2\dot{\varphi}, \tag{2c}$$

where $A_0 = Ae/mc$, $m = m_0(1 - v^2/c^2)^{-1/2}$ (in a stationary magnetic field $v^2 = \text{const}$). System (2) is nonlinear, and it is not possible to obtain its general solution in analytic form. In (^{2}) a number of trajectories obtained by numerical integration are presented. We shall show that, in addition to the trivial solutions $\rho = \dot{\rho} = 0$ and $z = \dot{z} = 0$, corresponding to uniform motion along the z-axis and in the radial direction in the plane of symmetry of the trap $z = 0$ (when the Lorentz force is equal to zero), system (2) admits one more exact particular solution, which corresponds to motion along the surface of the cone $\rho^2 = z^2$*.

Integrating equation (2b), we have

$$\rho^2\dot{\varphi} + A_0 z\rho^2 = \text{const}. \tag{3}$$

For trajectories passing through the origin, equation (3) gives

$$\dot{\varphi} = -A_0 z, \tag{4}$$

if $\rho \neq 0$. Obviously, (4) is also valid at the origin, since as $z \to 0$, $\dot{\varphi} \to 0$. Substituting (4) into (2a) and (2c), we see that the equations of motion (2) are compatible with the condition $\rho^2 = z^2$. Consequently, in the magnetic field (1) there is a family of trajectories lying on the surface of the cone $\rho^2 = z^2$ (it is not difficult to prove that trajectories not passing through the origin cannot lie on this cone). For $\rho^2 = z^2$, equations (2a) and (2c) reduce to the equation

$$\ddot{z} + A_0^2 z^3 = 0. \tag{5}$$

Let at $t = 0$, $z = 0$, $\dot{z} = \pm v_1$ ($v_1 > 0$). Integrating (5), we obtain

$$\dot{z}^2 = v_1^2 - A_0^2 z^4/2. \tag{6}$$

Putting $\dot{z} = 0$, we determine from (6) the position of the reflection point

$$z_{\max}^2 = \rho_{\max}^2 = \sqrt{2}\, v_1/A_0. \tag{7}$$

* In the two-dimensional problem ($H_x = Ay$; $H_y = Ax$; $H_z = 0$), exact particular solutions of the equations of motion in the planes $x = 0$ and $y = 0$ were obtained in work ([3]).

According to (6),

$$t = \pm \frac{1}{v_1} \int_0^z \left(1 - \frac{A_0^2 z^4}{2 v_1^2}\right)^{-1/2} dz.$$

Setting $\sin x = (A_0/\sqrt{2} v_1)^{1/2} z = z/z_{\max}$ and integrating, we find (see (4))

$$(\sqrt{2} A_0 v_1)^{1/2} t = \pm F(\varphi, k), \tag{8}$$

where $F(\varphi, k)$ is an elliptic integral of the first kind; the modulus is $k = 1/\sqrt{2}$; $\sin\varphi = \sqrt{2} \sin x (1 + \sin^2 x)^{-1/2} = \sqrt{2} z (z_{\max}^2 + z^2)^{-1/2}$. It is convenient to pass to Jacobi functions. Then (8) can be rewritten in the form

$$z = \pm z_{\max} \frac{\operatorname{sn}\left(\sqrt{\sqrt{2} A_0 v_1}\, t\right)}{\left[2 - \operatorname{sn}^2\left(\sqrt{\sqrt{2} A_0 v_1}\, t\right)\right]^{1/2}}, \tag{9}$$

where $z_{\max}$ is determined by (7), sn is the elliptic sine, and the plus and minus signs in (8) and (9) correspond to the different initial conditions $\dot{z}(t=0) = \pm v_1$. From equation (4) we have

$$d\varphi = -A_0 z\, dz/\dot{z}, \tag{10}$$

where $\dot{z}$ is determined by expression (6). Put $\varphi(t=0) = \varphi_0$.* Integrating (10), we obtain (see (4))

$$\varphi = \varphi_0 \mp \sqrt{2}\,\frac{\pi}{4} \pm \sqrt{2}\arcsin \frac{\operatorname{cn}\left(\sqrt{\sqrt{2}A_0 v_1}\, t\right)}{\left[2 - \operatorname{sn}^2\left(\sqrt{\sqrt{2}A_0 v_1}\, t\right)\right]^{1/2}}, \tag{11}$$

where cn is the elliptic cosine, and the upper sign corresponds to the initial condition $\dot{z}(t=0) = +v_1$.

Let us determine the time interval T during which the particle travels from the vertex of the cone to $z = z_{\max}$. Since $\operatorname{sn} K = 1$, it follows from (9) that

$$T = K(1/\sqrt{2})/(\sqrt{2}A_0 v_1)^{1/2}, \tag{12}$$

where

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 = 1.8541$$

is the complete elliptic integral of the first kind with modulus $k = 1/\sqrt{2}$.

Expressions (9), (11), and (12) make it possible to trace the trajectory of the particle over the course of the full period of its motion. Let, for definiteness, $\dot{z}(t=0) = -v_1$. Using the periodicity properties of the Jacobi functions, we obtain: at $t = T$, $\varphi = \varphi_0 + \sqrt{2}\pi/4$, $z = -z_{\max}$; at $t = 2T$, $\varphi = \varphi_{\max} = \varphi_0 + \sqrt{2}\pi/2$, $z = 0$; at $t = 3T$, $\varphi = \varphi_0 + \sqrt{2}\pi/4$, $z = z_{\max}$; finally, at $t = 4T$, $\varphi = \varphi_0$, $z = 0$. Thus, the trajectory of a particle moving on the surface of the cone $\rho^2 = z^2$ resembles a figure eight, and the complete period of the motion is equal to $4T$.

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* Obviously, φ_0 is the angle between the projection of the particle velocity vector onto the plane $z = 0$ and the x-axis at $t = 0$. Indeed, differentiating the relations $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ with respect to time. Since, according to (4), $\dot{\varphi}(t = 0) = -A_0 z(t = 0) = 0$, we have $\tan\varphi_0 = \dot{y}(t = 0)/\dot{x}(t = 0)$.

Note: Figure translations are in progress. See original paper for figures.

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