THE CHARACTERISTIC EQUATION OF THE THEORY OF RADIATION TRANSFER

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.947.32

MATHEMATICS

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THE CHARACTERISTIC EQUATION OF THE THEORY OF RADIATION TRANSFER

(Presented by Academician M. V. Keldysh on 16 IX 1965)

The problem of determining the asymptotics of the spatial-angular distribution of radiation in the depth of a thick plane layer of matter reduces to the study of the equation ([1])

$$(1 + k\vec{\omega}\mathbf{n})\Phi(\vec{\omega}) = \hat{g}\Phi(\vec{\omega}). \tag{1}$$

Here k is a parameter, $\vec{\omega}$ is a vector running over the unit sphere Ω of three-dimensional Euclidean space; $\mathbf{n}$ is a fixed vector from Ω; $\hat{g}$ is an integral operator:

$$\hat{g}f(\vec{\omega}) = \int_{\Omega} g(\overrightarrow{\omega\omega'}) f(\overrightarrow{\omega'})\, d\omega'.$$

The kernel $g(\overrightarrow{\omega\omega'})$ is determined by specifying the scattering indicatrix $g(\mu)$, $\mu \in [-1, 1]$, and depends on the scalar product $\overrightarrow{\omega\omega'}$ of the vectors $\vec{\omega}$ and $\overrightarrow{\omega'}$ from Ω;

$$\int_{\Omega} \cdots\, d\omega'$$

is the integral with respect to Lebesgue measure on Ω.

We shall be interested in the question of for which values of k there exist, and how the nontrivial solutions of (1) are arranged; and how the solutions of the nonhomogeneous equation corresponding to (1) depend on k.

We shall assume that

$$1) \quad g \in L_2(\Omega); \qquad g(\mu) \geqslant 0 \text{ on } [-1, 1]; \qquad 0 < 2\pi \int_{-1}^{1} g(\mu)\, d\mu \leqslant 1.$$

This condition makes it possible to expand $g(\mu)$ in a Legendre-polynomial series convergent in the mean:

$$g(\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} g_n P_n(\mu), \qquad 0 < g_0 \leqslant 1,$$

$$|g_n| < g_0 \quad \text{for } n = 1, 2, \dots .$$

Let $L_2(\Omega)$ and $C(\Omega)$ be the spaces of functions on Ω that are respectively square-summable and continuous;

$$(f_1, f_2) = \int_\Omega f_1(\vec{\omega}) \overline{f_2(\vec{\omega})}\, d\omega$$

is the scalar product in $L_2(\Omega)$;

$$\|f\| = (f, f)^{1/2}, \quad f \in L_2(\Omega),$$

and

$$\|f\|_{C(\Omega)} = \sup_{\vec{\omega} \in \Omega} |f(\vec{\omega})|, \quad f \in C(\Omega).$$

Denote by ϑ the zero of $L_2(\Omega)$, and by Z_0 the complex k-plane with cuts along the real axis from $-\infty$ to -1 and from 1 to ∞.

Theorem 1. $\hat{g}$ is a linear bounded self-adjoint operator mapping $L_2(\Omega)$ into itself, with norm $\|\hat{g}\| = g_0$. $\hat{g}(L_2(\Omega)) \subset C(\Omega)$; for $f \in L_2(\Omega)$

$$\|\hat{g} f\|_{C(\Omega)} \leqslant \|f\| \left(2\pi \int_{-1}^{1} g^2(\mu)\, d\mu \right)^{1/2};$$

as an operator acting from $L_2(\Omega)$ to $C(\Omega)$, $\hat{g}$ is completely continuous.

Define, for $k \in Z_0$,

$$\hat{U}(k) f(\vec{\omega}) = (1 + k\vec{\omega}\mathbf{n})^{-1} \hat{g} f(\vec{\omega}).$$

$$\hat{U}(k) : L_2(\Omega) \to$$

$\to L_2(\Omega)$, $\hat{U}(k)$ is completely continuous. Let $\sigma(\hat{U}(k))$ be its spectrum. Obviously, (1) is solvable if and only if $1 \in \sigma(\hat{U}(k))$.

It can be shown that for $k \in (-1, 1)$ and sufficiently large p

$$\inf_{\vec{\omega} \in \Omega} \hat{U}^p(k) f(\vec{\omega}) > 0$$

for every nonnegative $f(\vec{\omega})$. Relying on the theory of cones in Banach spaces ([2]), from this we obtain:

Theorem 2. Let $k \in (-1, 1)$, $M(k) = \sup\{|\lambda| \mid \lambda \in \sigma(\hat{U}(k))\}$. Then:

I. $M(k) > 0,\ M(k) \in \sigma(\hat{U}(k))$.

II. If $\lambda \in \sigma(\hat{U}(k))$, $\lambda \neq M(k)$, then $|\lambda| < M(k)$.

III. $M(k)$ is a simple eigenvalue of $\hat{U}(k)$.

IV. There exists exactly one function $\Phi_k \in L_2(\Omega)$ possessing the following properties:

$$\hat{U}(k)\Phi_k = M(k)\Phi_k, \qquad (\Phi_k, 1) = 4\pi.$$

Moreover,

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$$\inf_{\vec{\omega} \in \Omega} \Phi_k(\vec{\omega}) > 0.$$

VI. If $x \neq \vartheta$, $x \in C(\Omega)$, $x(\vec{\omega}) \geq 0$ on Ω, and x is an eigenfunction of the operator $\hat{U}(k)$, then there exists $a > 0$ such that $x = a\Phi_k$.

In order to prove the existence of nontrivial solutions of (1), it is now sufficient to establish the solvability of the equation $M(k) = 1$ in the interval $-1 < k < 1$.

Let us note that $\hat{U}(k)$ is an analytic function for $k \in Z_0$, and for $k_0 \in (-1, 1)$, according to Theorem 2, $M(k_0)$ is a simple isolated eigenvalue of $\hat{U}(k_0)$. This permits one to apply to the analysis of the behavior of $M(k)$ and Φ_k in a neighborhood of the point k_0 the theory of analytic perturbations of operators (for example ([6])), which gives:

Theorem 3. There exist an open set $G_0 \subset Z_0$, a numerical function $\lambda(k)$, and a function $e(k)$ with values in $L_2(\Omega)$, defined on G_0, such that:

I. $(-1, 1) \subset G_0$.

II. $\lambda(k)$ and $e(k)$ are holomorphic in G_0.

III. For $k \in G_0$, $\|e(k)\| > 0$, $\lambda(k) \neq 0$, $\hat{U}(k)e(k) = \lambda(k)e(k)$.

IV. For $k \in (-1, 1)$, $\lambda(k) = M(k)$, $e(k) = \Phi_k$.

According to Theorem 3, in a neighborhood of each point $k_0 \in G_0$, $e(k)$ is expanded in a power series in $(k - k_0)$, convergent in the metric of $L_2(\Omega)$. It turns out that the coefficients of this series are elements of $C(\Omega)$, and the series itself, for sufficiently small $(k - k_0)$, converges uniformly with respect to $\vec{\omega} \in \Omega$. This, in particular, means that for $k_1 \in (-1, 1)$, uniformly with respect to $\vec{\omega} \in \Omega$,

$$\lim_{k=k_1} \Phi_k(\vec{\omega}) = \Phi_{k_1}(\vec{\omega}).$$

Using the special form of the kernel of the operator $\hat{g}$ and the analyticity of $M(k)$ and Φ_k, one can obtain the following results:

Theorem 4. If $k \in (-1, 1)$, then $M(-k) = M(k)$; $\Phi_k(-\vec{\omega}) = \Phi_{-k}(\vec{\omega})$ for $\vec{\omega} \in \Omega$; $M(0) = g_0$; $\Phi_0(\vec{\omega}) \equiv 1$ on Ω; $dM(k)/dk > 0$ for $k \in (0, 1)$; $dM(k)/dk|_{k=0} = 0$; $\lim_{k=1-0} M(k) = \infty$. There exists exactly one $\lambda_0 \in [0, 1)$ such that $M(\lambda_0) = 1$. If $g_0 = 1$, then $\lambda_0 = 0$; if $g_0 < 1$, then $\lambda_0 > 0$.

Theorem 5. Let $k \in (-1, 1)$, $x \in L_2(\Omega) \setminus \{\vartheta\}$,

$$(1 + k\vec{\omega}\mathbf{n})x(\vec{\omega}) = \hat{g}x(\vec{\omega}).$$

I. If $x(\vec{\omega}) \geq 0$ on Ω, then either $k = \lambda_0$ and $x = a_1\Phi_{\lambda_0}$, or $k = -\lambda_0$ and $x = a_2\Phi_{-\lambda_0}$, $(a_{1,2} = \text{const} > 0)$.

II. If $|k| \neq \lambda_0$ and $x(\vec{\omega})$ is a real function, then $|k| > \lambda_0$ and $x(\vec{\omega})$ assumes on Ω both positive and negative values.

Denote by $\mathfrak{N}$ the set of those $k \in Z_0$ for each of which (1) admits a nontrivial solution $\Phi \in L_2(\Omega)$. Then $\mathfrak{N} = \mathfrak{N}_0 \cup (-\mathfrak{N}_0)$, $\mathfrak{N}_0 \subset [\lambda_0, 1)$, $\lambda_0 \in \mathfrak{N}_0$, and $\mathfrak{N}_0$ is at most countable. If $\mathfrak{N}_0$ is infinite, then 1 is the only limit point of $\mathfrak{N}_0$. For $k \in Z_0$ define the linear bounded operator $\widehat{W}(k)$, acting in $L_2(\Omega)$, by the rule:

$$\widehat{W}(k)x = (1 + k\vec{\omega}\mathbf{n})x - \hat{g}x, \qquad x \in L_2(\Omega).$$

If $k \in Z_0 \setminus \mathfrak{N}$, then $\widehat{W}(k)$ implements a one-to-one mapping of $L_2(\Omega)$ onto itself. For $k \in \mathfrak{N}$ put

$$H_k = \{x \mid x \in L_2(\Omega),\ \widehat{W}(k)x = 0\}.$$

Then H_k is a finite-dimensional subspace of $L_2(\Omega)$. Let p_k be the dimension of H_k. Then $p_{\pm\lambda_0} = 1$, and for all $k \in \mathfrak{N}$, $p_k = p_{-k} \geqslant 1$. In the subspaces H_k, with $k \in \mathfrak{N}$, $k \neq 0$, bases $\{\psi_{kp} \mid p = 1, 2, \dots, p_k\}$ may be chosen in such a way that: a) $\psi_{kp}(\vec{\omega})$ is a real continuous function of $\vec{\omega} \in \Omega$; b) $(\vec{\omega}\mathbf{n}, \psi_{kp}^2) = -\operatorname{sgn} k$ for all $p = 1, 2, \dots, p_k$, and $((\vec{\omega}\mathbf{n})\psi_{kp}, \psi_{kp'}) = 0$ for $1 \leqslant p < p' \leqslant p_k$; c) for all $k \in \mathfrak{N}$, $k \neq 0$, and $p = 1, 2, \dots, p_k$,

$$\psi_{(-k)p}(\vec{\omega}) = \psi_{kp}(-\vec{\omega});$$

d) if $g_0 < 1$, then

$$\psi_{\lambda_0 1} = \left|(\vec{\omega}\mathbf{n}, \Phi_{\lambda_0}^2)\right|^{-1/2} \Phi_{\lambda_0}.$$

Let us now turn to the inhomogeneous equation corresponding to (1).

Theorem 6. *Let G be an open set, $G \subset Z_0$, and let $a(k)$ be a function with values in $L_2(\Omega)$, holomorphic in G. For each $k \in G \setminus \mathfrak{N}$ there exists exactly one solution $\psi = \psi(k) \in L_2(\Omega)$ of the equation*

$$(1 + k\vec{\omega}\mathbf{n})\psi(k) = \hat{g}\psi(k) + a(k).$$

This solution is $\psi(k) = W^{-1}(k)a(k)$. It is a single-valued analytic function of k in $G \setminus \mathfrak{N}$. Let $k_0 \in G \cap \mathfrak{N}$. Then there exists $r > 0$ such that

$$S(k_0, r) \equiv \{k \mid |k - k_0| < r\} \subset G, \qquad S(k_0, r) \cap \mathfrak{N} = \{k_0\}.$$

In $S(k_0, r) \setminus \{k_0\}$ the solution $\psi(k)$ can be represented as follows:

a) *if $g_0 = 1$ and $k_0 = 0$, then*

$$\psi(k, \vec{\omega}) = -\frac{3}{4\pi}\frac{1 - g_1}{k^2}(a(0), 1) + \frac{3}{4\pi k}\left[(a(0), 1)\vec{\omega}\mathbf{n} - (1 - g_1)(a'(0), 1) + (a(0), \vec{\omega}\mathbf{n})\right] + u(k, \vec{\omega});$$

b) *if $k_0 \neq 0$, then*

$$\psi(k, \vec{\omega}) = -\frac{\operatorname{sgn} k}{k - k_0}\sum_{p=1}^{p_{k_0}}(a(k_0), \psi_{k_0 p})\psi_{k_0 p}(\vec{\omega}) + u(k, \vec{\omega}).$$

In both cases $u(k)$ is a function of k with values in $L_2(\Omega)$, holomorphic in $S(k_0, r)$.

Let us return to the operator $\widehat{U}(k)$. It turns out that separation of the polar and azimuthal coordinates of the vector $\vec{\omega}$ occurs in all eigenfunctions of $\widehat{U}(k)$. In order to describe this phenomenon accurately, we introduce the following notation. Let

$$\Delta = [-1, 1] \times [0, 2\pi]$$

be a rectangle in the (μ, φ)-plane, and let σ be an arbitrary orthonormal basis of three-dimensional space with third vector $\mathbf{n}$:

$$\sigma = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}.$$

Let

$$\vec{\psi}_\sigma(\mu, \varphi) = \sqrt{1 - \mu^2}\cos\varphi\,\mathbf{e}_1 + \sqrt{1 - \mu^2}\sin\varphi\,\mathbf{e}_2 + \mu\mathbf{n}, \qquad (\mu, \varphi) \in \Delta,$$

and let $f \in C(\Omega)$. Fix $\vec{\omega} \in \Omega$ and $(\mu, \varphi) \in \vec{\psi}_\sigma^{-1}(\vec{\omega})$. For each integer s, put

$$\widehat{P}^{(s)} f(\vec{\omega}) = \frac{1}{2\pi}e^{is\varphi}\int_0^{2\pi} e^{-is\varphi'} f(\vec{\psi}_\sigma(\mu, \varphi'))\,d\varphi'.$$

It turns out that $\widehat{P}^{(s)}(C(\Omega)) \subset C(\Omega)$, and moreover the $\widehat{P}^{(s)}$ can be extended by continuity to all of $L_2(\Omega)$. As an operator acting in $L_2(\Omega)$, each $\widehat{P}^{(s)}$ is a projection operator. $\widehat{P}^{(s)}\widehat{P}^{(s')} = \delta_{ss'}\widehat{P}^{(s)}$ for all integers s and s'. Finally, each $\widehat{P}^{(s)}$ commutes with $\hat{g}$.

We now note that the function of three variables (μ, μ', φ')

$$g\left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\varphi'\right)$$

is measurable and square-summable on the parallelepiped $[-1,1] \times [-1,1] \times [0, 2\pi]$. This makes it possible to define, for integral s, functions $g_s(\mu, \mu')$ from $L_2([-1,1] \times [-1,1])$ by the formula

$$g_s(\mu, \mu') = \int_0^{2\pi} g\left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\varphi'\right) e^{is\varphi'}\, d\varphi'.$$

Let us associate with each g_s the operator $\hat{g}_s$ acting in $L_2(-1,1)$:

$$\hat{g}_s f(\mu) = \int_{-1}^{1} g_s(\mu, \mu')\, f(\mu')\, d\mu' \quad \text{for } f \in L_2(-1,1).$$

Theorem 7. Let $k \in Z_0$ and $\lambda \in \sigma(\hat{U}(k))$, $\lambda \neq 0$. If $x \in L_2(\Omega)$, $\hat{U}(k)x = \lambda x$, then among the functions $x_s = \hat{P}^{(s)}x$ there is only a finite number of functions x_{s_ν}, $\nu = 1, 2, \ldots, \nu_0$, distinct from zero. Moreover,

$$\hat{U}(k)x_{s_\nu} = \lambda x_{s_\nu} \quad \text{and} \quad x = \sum_{\nu=1}^{\nu_0} x_{s_\nu}.$$

There exist functions $a_\nu \in L_2(-1,1)$ such that for all $\vec{\omega} \in \Omega$

$$x_{s_\nu}(\vec{\omega}) = a_\nu(\mu)e^{is_\nu\varphi}, \quad (\mu, \varphi) \in \vec{\psi}_\sigma^{-1}(\vec{\omega}).$$

The functions $a_\nu(\mu)$ satisfy the equations

$$\lambda(1 + k\mu)a_\nu(\mu) = \hat{g}_{s_\nu} a_\nu(\mu), \quad \nu = 1, 2, \ldots, \nu_0,$$

and depend continuously on $\mu \in [-1, 1]$.

The converse is also true: if a_ν satisfy the equations from Theorem 7, then any linear combination of the corresponding x_{s_ν} is an eigenfunction of $\hat{U}(k)$ belonging to λ.

Theorem 8. Let $k \in (-1, 1)$. For all integral s,

$$\hat{P}^{(s)}\Phi_k = \delta_{s0}\Phi_k.$$

There exists exactly one function $\varphi_k(\mu)$ of the variable $\mu \in [-1,1]$, $\varphi_k \in L_2(-1,1)$, such that, for $\vec{\omega} \in \Omega$,

$$\Phi_k(\vec{\omega}) = \varphi_k(\vec{\omega}\mathbf{n}).$$

Moreover,

$$\varphi_k \in C(-1,1), \qquad M(k)(1+k\mu)x(\mu) = \hat{g}_0 x(\mu), \qquad \int_{-1}^{1} \varphi_k(\mu)\, d\mu = 2.$$

The equations

$$M(k)(1+k\mu)\varphi_k(\mu) = \hat{g}_s \varphi_k(\mu)$$

for $s \neq 0$ admit only the trivial solution $x(\mu) \equiv 0$.

The connection of φ_k with Φ_k makes it possible to elucidate many properties of φ_k. Deeper results can be obtained by a direct study of the equation for φ_k ([3–5]).

Received
14 IX 1965

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Note: Figure translations are in progress. See original paper for figures.