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Abstract

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ON THE REGULARIZATION OF THE SUPREMUM OF A FAMILY OF PLURISUBHARMONIC FUNCTIONS AND ITS APPLICATION TO ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

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Let $\{U_\alpha(P)\}$ be a certain family of functions, subharmonic and uniformly bounded above in a domain G of the m-dimensional space R^n. Let, further,

$$U(P) = \sup_\alpha U_\alpha(P).$$

If the family under consideration consists of a finite number of functions, then, as is known (see, for example, (1)), $U(P)$ is a subharmonic function. In the case of an infinite number of functions this assertion is false, since in this case the function $U(P)$ need not be upper semicontinuous. However, the least upper semicontinuous majorant of the function $U(P)$ will in this case also be a subharmonic function (see, for example, (2)).

It is easy to see that the function $\widetilde{U}(P)$, which we shall call the regularization of the function $U(P)$, is defined by the relation

$$\widetilde{U}(P) = \varlimsup_{P' \to P} U(P').$$

A. Cartan (3) proved the following theorem, from which it follows that the function $U(P)$ is, in a certain sense, almost subharmonic.

Theorem (A. Cartan). *Let $\{U_\alpha(P)\}$ be a certain family of functions, subharmonic and uniformly bounded in a domain $G \subset R^n$. Then everywhere, with the possible exception of a set of outer capacity zero, the equality*

$$U(P) = \widetilde{U}(P)$$

holds.

In this note we extend Cartan' s theorem to plurisubharmonic functions*, considering, along with the regularization of the supremum, the regularization of the upper limit. The results obtained are then applied to the study of functions analytic in Hartogs domains**.

We introduce the following notation.

The space of complex variables $z_1, \dots, z_n$ will be denoted by C^n.

By $C(E)$, where E is a certain plane set, we shall denote the inner capacity of the set E.

By $\Delta(E; z'_1, \dots, z'_{k-1})$, where the set $E \subset C^k$, we shall denote the intersection of E with the plane

$$\{(z_1, \dots, z_k) : z_1 = z'_1, \dots, z_{k-1} = z'_{k-1},\ |z_k| < \infty\}.$$

For $k > 1$, by $\Gamma_k(E)$, where the set $E \subset C^k$, we shall denote the set of those points $(z_1, \dots, z_{k-1})$ for which $C(\Delta(E; z_1, \dots, z_{k-1})) > 0$.

* An upper semicontinuous real function $U(z_1, \dots, z_n)$ is called plurisubharmonic if, for any complex a_i and b_i, $i = 1, 2, \dots, n$, the function $U(a_1\xi + b_1, \dots, a_n\xi + b_n)$ is a subharmonic function of the variable ξ. For more details see, for example, (^{2}).

** On Hartogs domains see, for example, (^{2},^{4},^{5}).

We also put $\Gamma_1(E) = E$ and $\Gamma_k^1(E) = \Gamma_1(\Gamma_2(\dots(\Gamma_k(E))\dots))$. Let $U(z_1, \dots, z_n)$ be a certain real-valued function in a domain $G \subset C^n$. Denote

$$\widetilde{U}(z_1, \dots, z_n) = \lim_{\varepsilon \to 0} \sup_{|z_i - z'_i| < \varepsilon,\ i=1,\dots,n} U(z'_1, \dots, z'_n).$$

Theorem 1. Let $\{U_\alpha(z_1, \dots, z_n)\}$ be a family of plurisubharmonic functions, uniformly bounded above in some domain $G \subset C^n$. Let, further,

$$U(z_1, \dots, z_n) \overset{\text{def}}{=} \sup_\alpha U_\alpha(z_1, \dots, z_n).$$

Then the set E_U of those points $(z_1, \dots, z_n) \in G$ at which the inequality

$$U(z_1, \dots, z_n) < \widetilde{U}(z_1, \dots, z_n)$$

holds belongs to $G_{\delta\sigma}$ and satisfies the condition $C(\Gamma_n^1(E_U)) = 0$.

Theorem 2. Let $\{U_\alpha(z_1, \dots, z_n)\}$, where $1 \le \alpha < \infty$, be a family of plurisubharmonic functions, uniformly bounded above in some domain $G \subset C^n$. Let, further,

$$V(z_1, \dots, z_n) \overset{\text{def}}{=} \varlimsup_{\alpha \to \infty} U_\alpha(z_1, \dots, z_n).$$

Then the set E_V of points $(z_1, \dots, z_n) \in G$ at which the inequality

$$V(z_1, \dots, z_n) < \widetilde{V}(z_1, \dots, z_n)$$

holds belongs to $G_{\delta\sigma}$ and satisfies the condition $C(\Gamma_n^1(E_V)) = 0$.

Let us note that plurisubharmonic functions are at the same time subharmonic. Therefore the theorem of H. Cartan cited at the beginning of the article is applicable to them. However, the characterization of the sets E_U and E_V thereby obtained will be somewhat coarser than that given in Theorems 1 and 2. Indeed, it is not hard to see that the set

$$E = \{(z_1, z_2) : \operatorname{Im} z_1 = 0,\ \operatorname{Im} z_2 = 0,\ 0 \le \operatorname{Re} z_1 \le 1,\ 0 \le \operatorname{Re} z_2 \le 1\}$$

has zero capacity in the space of the variables $\operatorname{Im} z_1, \operatorname{Re} z_1, \operatorname{Im} z_2, \operatorname{Re} z_2$, whereas $C(\Gamma_2^1(E)) > 0$.

Let us also note that the functions $U(z_1, \dots, z_n)$ and $V(z_1, \dots, z_n)$ occurring in Theorems 1 and 2 are plurisubharmonic, except in the case $V(z_1, \dots, z_n) \equiv -\infty$.

We shall apply Theorem 2 to the study of functions analytic in Hartogs domains, i.e. in domains of the form

$$G_{D,\varphi} = \{(z_1, \dots, z_n, w) : (z_1, \dots, z_n) \in D,\ |w| < \varphi(z_1, \dots, z_n)\},$$

where D is some domain in C^n, and the function $\varphi(z_1, \dots, z_n)$ is defined in the domain D, with $\varphi(z_1, \dots, z_n) > 0$ everywhere in D.

Theorem 3. Let the function $f(z_1, \dots, z_n, w)$ be holomorphic in some domain $G_{D,\varphi}$ and not holomorphic in any domain $G_{D,\psi}$ with a function $\psi(z_1, \dots, z_n)$ satisfying, at least at one point $(z_1^0, \dots, z_n^0) \in D$, the inequality

$$\varphi(z_1^0, \dots, z_n^0) < \psi(z_1^0, \dots, z_n^0) * .$$

Let, further, E_f be the set of those points $(z_1, \dots, z_n) \in D$ for which the function $f(z_1, \dots, z_n, w)$, considered as a function of the single variable w, is holomorphic in some disk $|w| < R(z_1, \dots, z_n)$ with

$$R(z_1, \dots, z_n) > \varphi(z_1, \dots, z_n).$$

Then $E_f \in F_\sigma$ and $C(\Gamma_n^1(E_f)) = 0$.

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* In the case under consideration, $\varphi(z_1, \dots, z_n)$ is called the radius of regularity, and $-\ln \varphi(z_1, \dots, z_n)$ is a plurisubharmonic function.

Proof. As is known (see, for example, ([2],[4] ,[5])), the function $f(z_1, \dots, z_n, w)$ can be expanded in the domain $G_{D,\varphi}$ into the series

$$f(z_1, \dots, z_n, w) = \sum_{k=0}^{\infty} w^k g_k(z_1, \dots, z_n).$$

Here the function $\varphi(z_1, \dots, z_n)$ is related to the coefficients of the series by the relation

$$\frac{1}{\varphi(z_1, \dots, z_n)} = \lim_{\varepsilon \to 0} \sup_{|z_i - z_i'| < \varepsilon,\ i=1,\dots,n} \left\{ \overline{\lim_{k\to\infty}} \sqrt[k]{|g_k(z_1, \dots, z_n)|} \right\},$$

and the points $(z_1, \dots, z_n) \in E_f$ are determined by the equality

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|g_k(z_1, \dots, z_n)|} < \frac{1}{\varphi(z_1, \dots, z_n)}.$$

It is not difficult also to see that the sequence of continuous plurisubharmonic functions $\left\{ \sqrt[k]{|g_k(z_1, \dots, z_n)|} \right\}$ is uniformly bounded on every compact set $K \subset D$, $E_f \subset G_{\delta\sigma}$. Consequently, to prove Theorem 3 it now suffices to refer to Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.

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