E. Shamir (E. Shamir)

1966

SovietRxiv

Abstract

Full Text

E. Shamir (E. Shamir)

SOLUTIONS OF RIEMANN-HILBERT SYSTEMS WITH PIECEWISE-CONTINUOUS COEFFICIENTS IN L^p

(Presented by Academician L. S. Pontryagin, 27 X 1965)

§ 1. Let $P(x)$ be an $m \times m$ matrix function such that:

a) $P(x)$ is piecewise continuously differentiable on the extended real line and has discontinuities (jumps) at the points $b_1, \dots, b_n$ ($b_n = \infty$, if $P(-\infty) \neq P(\infty)$).

b) $P(x)$ is a nonsingular matrix at every point of its domain of definition (including $x = b_r \pm 0,\ \pm\infty$).

c)

$$\left(\frac{d}{dx}\right)^k [P(x) - P(\pm\infty)] = O(|x|^{-k}), \quad x \to \pm\infty, \quad k = 1, 2.$$

Consider the following Riemann-Hilbert problem:

Find a pair of vector functions $\Phi^{\pm}(x)$ with components in $L^p,\ 1 < p < \infty$ (we shall simply write $\Phi^{\pm} \in L^p$), admitting holomorphic extensions to $\pi_{\pm}$ (respectively the upper and lower half-planes of the complex plane) and satisfying the condition

$$\Phi^-(x) - P(x)\Phi^+(x) = g(x), \quad x \text{ real}, \tag{1}$$

where g is given in L^p.

This problem is equivalent to solving (in L^p) the system of singular integral equations

$$\mathbf{P}f \equiv [H_- - P(x)H_+]f = g, \tag{2}$$

where

$$(H_{\pm}f)(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(y)}{x \pm i0 - y}\, dy. \tag{3}$$

The set of all $g \in L^p$ for which system (1) is solvable is called the range of the problem (in L^p). This is in fact the range of $\mathbf{P}$.

The Riemann-Hilbert problem (see ([1,2,4,5])) is usually solved by representing $P(x)$ in the form $P_-^{-1}(x)P_+(x)$, where $P_\pm$ are holomorphic in $\pi_\pm$. In the case $P(x) = I + k^\wedge(x)$ (I is the identity matrix and $k^\wedge$ is the Fourier transform of k), the corresponding factorization was considered by Gohberg and Krein in ([1]). In this case $P(x)$ is uniformly continuous and $P(\pm\infty) = I$. Our method consists in "filling in" the jumps of $P(x)$ by factors holomorphic in $\pi_\pm$, and then factorizing as in Gohberg and Krein. Once the factorization of P is known, system (1) is transformed into a diagonal system of a certain type, which is solvable explicitly. Using for $H_\pm$ certain weighted estimates in L^p, considered in ([3,5]), we show that the solutions belong to L^p, except for some precisely determined cases. The scalar case ($m = 1$) was essentially considered in ([5]) by Widom.

§ 2. Consider a point of discontinuity $b_r \neq \infty$ (the case $b_r = \infty$ is somewhat different). Two cases are possible. Either the jump at b_r

$$P(b_r - 0)P^{-1}(b_r + 0)$$

is similar to a diagonal matrix

$$P(b_r - 0)P^{-1}(b_r + 0) \sim \operatorname{diag}[\lambda_{r1}, \dots, \lambda_{rm}] \tag{4}$$

(the first case), or it is not (the second case).

In the first case, let

$$-\frac{\log \lambda_{rj}}{2\pi i} = \zeta_{rj} = \sigma_{rj} + i\tau_{rj}, \qquad -1/p' \leq \sigma_{rj} < 1/p, \qquad 1 \leq j \leq m, \tag{5}$$

where $1/p' + 1/p = 1$. Let

$$\Gamma_r^\pm = \operatorname{diag}\left[\left(\frac{x - b_r \pm i0}{x - b_r \pm i}\right)^{\zeta_{r1}}, \dots, \left(\frac{x - b_r \pm i0}{x - b_r \pm i}\right)^{\zeta_{rm}}\right] \tag{6}$$

the exponents being determined by the principal values of the logarithm in the plane cut along the negative axis. It is easy to see that the function

$$P_1(x) = B\Gamma_r^-(x)AP(x)P^{-1}(b_r + 0)A^{-1}\Gamma[\Gamma_r^+(x)]^{-1} \tag{7}$$

for suitable constant matrices A and B is continuous at the point $x = b_r$ (by Tëlder), behaves at infinity analogously to P, and has the same smoothness as P at points $x \neq b_r$. In particular, the jumps of P and P_1 at a point of discontinuity

$b_s \neq b_r$ are similar matrices. This should be noted, since the nonconstant correcting factors $\Gamma_r^{\pm}(x)$ at the point b_r depend only on the similarity class in (4). Thus, after filling in all n jumps (the order is essentially immaterial), we obtain a continuous function P_n. From conditions a)-c) it further follows that the function P_n can be factorized as in (1). Indeed, for this it is enough to suppose that $P_n = I + k_n(x)$, $k(x) \in L^1$.

Remark 1. In the second case the factors $\Gamma_r^{\pm}$ become more complicated because of the addition of triangular matrices whose elements are powers of $\log(x - b_r \pm i0) \cdot [\log(x - b_r \pm i)]^{-1}$, but the main results remain valid.

Theorem 1. Let $P(x)$ satisfy conditions a), b), c). Then

$$P(x) = Q_-^{-1}(x)\,\mathrm{diag}\left[\left(\frac{x-i}{x+i}\right)^{k_1}, \dots, \left(\frac{x-i}{x+i}\right)^{k_m}\right] Q_+(x), \tag{8}$$

where

$$Q_{\pm}(x) = N^{\pm}(x) A_n^{\pm} \Gamma_n^{\pm}(x) A_{n-1}^{\pm} \Gamma_{n-1}^{\pm}(x) \dots \Gamma_1^{\pm}(x) A_0^{\pm}. \tag{9}$$

Here $A_r^{\pm}$ are certain constant nonsingular matrices, $\Gamma_r^{\pm}$ are given by the expressions (6) (taking Remark 1 into account). $N^{\pm}(x)$ are the same as in (1), i.e. they (and their inverse functions) are holomorphic in $\pi_{\pm}$, bounded and continuous in the closure of $\pi_{\pm}$. Finally, $k_1, \dots, k_m$ are integers ("proper exponents" in (1)) and

$$\sum_j k_j = \frac{1}{2\pi}\left\{\left(\int_{-\infty}^{b_1-0} + \int_{b_1+0}^{b_2-0} + \dots + \int_{b_n+0}^{\infty}\right)(d_x \arg[\det P(x)]) + \sum_r \sum_j \sigma_{rj}\right\} \tag{10}$$

(σ_{rj} are given by the expression (5)).

§ 3. Under the same assumptions and notation as in Theorem 1, the following holds.

Theorem 2. Problem (1) has a closed range in L^p if and only if all $\sigma_{rj} \neq -1/p'$. The range is given by the conditions

$$\int [Q_- g]_j(y)(y-i)^{-l-1}\,dy = 0, \qquad 0 \le l < k_j, \quad k_j > 0, \quad 1 \le j \le m, \tag{11}$$

and has codimension

$$\beta = \sum_{k_j > 0} (k_j).$$

The explicit solution is given by the expressions

$$\Phi^- = Q_-^{-1} H_-(Q_- g),$$

$$\Phi^+ = Q_+^{-1} \operatorname{diag}\left[\left(\frac{x-i}{x+i}\right)^{k_1}, \dots, \left(\frac{x-i}{x+i}\right)^{k_m}\right] H_+(Q_- g). \qquad (12)$$

If $k_j > 0$, then, also in view of (11), we have

$$[Q_+ \Phi^+]_j = (x+i)^{k_j} H_+\big[(x-i)^{-k_j} Q_- g\big]_j. \qquad (13)$$

The general solution of the homogeneous system ($g = 0$) is given by the basis

$$\Phi^{\pm}_{lj} = Q_{\pm}^{-1} x^l (x \pm i)^{k_j} e_j, \qquad 0 \le l < -k_j, \quad k_j < 0, \quad 1 \le j \le m, \qquad (14)$$

where e_j is the vector with j-th component equal to 1 and all the others equal to 0. This basis has

$$\alpha = \sum_{k_j < 0} (-k_j)$$

elements. The index of the problem is

$$\alpha - \beta = -\sum_j k_j$$

and can be computed by means of (10).

Remark 2. Assertions concerning the closedness of the range of values of the problem and the value of the index for any piecewise-continuous function $P(x)$ satisfying b) are obtained by the method of approximation. For this one may consider the operator

$$\mathbf{P}_\lambda = \mathbf{P} - \lambda I = (1-\lambda) H_- - (P(x) - \lambda) H_+$$

and in an obvious way find the essential spectrum of $\mathbf{P}$ and the index of $\mathbf{P}_\lambda$. However, in order to characterize the resolvent set of $\mathbf{P}$, it is necessary to know the indices k_j connected with the factorization.

To prove Theorem 2, we use the factorization of $\mathbf{P}$ and rewrite (1) in the form

$$\psi^- = \operatorname{diag}\left[\left(\frac{x-i}{x+i}\right)^{k_1}, \dots, \left(\frac{x-i}{x+i}\right)^{k_m}\right]\psi^+ = h, \tag{15}$$

where

$$\psi^- = Q_-\varphi^-, \qquad \psi^+ = Q_+\varphi^+, \qquad h = Q_-g.$$

Equation (15) splits into m scalar problems, each of which is easily solved in explicit form, and this leads to formulas (11)–(14). It can be shown that

$$\left\|Q_{-1}^{-1}H_-Q_-g\right\|_{L^p} \leq K\|g\|_{L^p} \tag{16}$$

if and only if all $\sigma_{rj} \neq -1/p'$, and in this case φ^- (and φ^+) belong to L^p.

Remark 3. The method of reducing the discontinuous case to the continuous one by filling in the jumps is analogous to the method in the work of N. P. Vekua ([4]) (or ([2,5]) in the scalar case). The same is true with respect to some other methods and conclusions used in the present paper, as well as in ([1]). However (as noted in ([1])) the smoothness conditions on the coefficients can be weakened; in particular, Hölder continuity need not be required. Thus, for a particular solution of the problem, the only assumption required is the piecewise continuity of the coefficients (see also Remark 2). Moreover, it seems important to us to solve this problem by the factorization method (Theorem 1), since the indicated method, from a unified point of view, illuminates the approach to this and many other problems ([6]).

In the present paper we are especially interested in solutions in L^p. Therefore the main point is the study of the behavior of $Q_{-1}^{-1}H_-Q_-g$ near the points of discontinuity b_j, which makes it possible to establish (16). The classical approach

to the problem is mainly connected with the boundedness conditions for the coefficients near the discontinuity points. (We note that our limiting case $\sigma_{jr} = -1/p'$ is an analogue of "special ends" in ([2,4]).) The analysis that makes it possible to establish (16) can easily be used to study solutions in other spaces, in particular, in L^p spaces with a weight (with a weight near b_j).

Remark 4. Consider system (2) with the integral (in (3)) over the set E and x belonging to E. Setting $P(x) = I,\ x \notin E$, we obtain (since $H_- - H_+ = I$) an equivalent system on the entire line. We may further assume that E is a finite union of intervals (or half-lines). Then, in the same way, our system is, up to a factor, the "principal part" of the expression

$$M(x)f(x) + \frac{1}{\pi i}\int_E \frac{K(x,y)}{x-y} f(y)\,dy = g(x), \qquad x \in E.$$

California University
Berkeley, USA

Received
21 X 1965

REFERENCES

[1] I. Ts. Gokhberg, M. G. Krein, *UMN*, **13**, 2, 3 (1958); Amer. Math. Soc. Translation Ser. 2, **14**, N. Y., 1960, p. 217.
[2] N. I. Muskhelishvili, *Singular integral equations*, M.–L., 1946.
[3] E. Shamir, J. Analyse Math., **7**, 277 (1964).
[4] N. P. Vekua, *Systems of singular integral equations*, M.–L., 1950.
[5] H. Widom, Trans. Am. Math. Soc., **97**, 131 (1960).
[6] I. Ts. Gokhberg, *UMN*, **19**, 1, 71 (1964).

Note: Figure translations are in progress. See original paper for figures.