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Abstract

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M. Sh. BIRMAN, M. Z. SOLOMYAK

ON THE APPROXIMATION OF FUNCTIONS OF THE CLASSES W_p^α BY PIECEWISE-POLYNOMIAL FUNCTIONS

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In the present note, for functions of the classes W_p^α of S. L. Sobolev-L. N. Slobodetskii, the rate of approximation by piecewise-polynomial functions is studied. The approximations considered are uniform and in the mean (with respect to some measure). The order estimates obtained are exact and are derived in the study of a certain special method of approximation. The results are applied to obtaining new estimates of the singular numbers of integral operators. Both the approximation theorems themselves and the concrete method of approximation described below, as well as the estimates of singular numbers, find substantial applications in the theory of double operator integrals of Stieltjes ([1,2]). Here we shall indicate only that it is precisely the needs of this theory that dictate the somewhat special character of the estimates in Theorems 2-4, where independence of the measure is essential. As another application of the main results of the paper, we obtain an order-sharp estimate of the ε-entropy of the unit ball in W_p^α $(\alpha p > \nu)$ as a compactum in C.

1. We give the necessary notation and definitions. Below Q_ν is the unit cube of ν-dimensional Euclidean space. Let $p > 1,\ \alpha > 0$. $[\alpha]$ is the integer part of α and $\theta = \alpha - [\alpha]$. The class $W_p^\alpha(Q_\nu)$ consists of those functions $u(x) \in L_p(Q_\nu)$ for which the functional $N(u) = N(u; \alpha, p, Q_\nu)$ is finite, where $N^p(u)$ is defined, for integral α, by the first, and for nonintegral α, by the second of the following two expressions:

$$\sum_{|\omega|=\alpha} \int_{Q_\nu} |D^\omega u|^p \, dx; \qquad \sum_{|\omega|=[\alpha]} \int_{Q_\nu} \int_{Q_\nu} \frac{|D^\omega u(x) - D^\omega u(y)|^p}{|x-y|^{p\theta+\nu}} \, dx \, dy.$$

The norm in $W_p^\alpha(Q_\nu)$ is defined as the sum of the quantity $N(u)$ and the norm of $u(x)$ in $L_p(Q_\nu)$. Everywhere below it is assumed that $\alpha p > \nu$, so that functions from W_p^α are continuous: $W_p^\alpha(Q_\nu) \subset C(Q_\nu)$.

For $\nu = 1$ we shall also consider the classes V_β of functions of bounded β-variation $(\beta \geq 1)$, i.e. functions for which the quantity

$$\|u\|_{V_\beta} = |u(+0)| + \left[\sup \sum_{k=1}^{m} |u(x_k) - u(x_{k-1})|^\beta\right]^{1/\beta}, \tag{1}$$

is finite, where the least upper bound is taken over all finite systems of points $0 = x_0 < x_1 < \dots < x_m = 1$. The second term on the right-hand side of (1) will be denoted by $V(u;\beta)$.

Let Λ be some partition of the cube Q_ν into a finite number of cubes, $|\Lambda|$ the number of cubes in such a partition. Let the function $f(x)$ be such that in each of the cubes of the partition Λ it coincides with some polynomial of degree not exceeding l. We denote the set of all such functions by $P(\Lambda;l)$, and the union of all sets $P(\Lambda;l)$ for $|\Lambda| \le n$ by $P(n;l)$. In assertions concerning functions of the classes W_p^α,

everywhere $l = \alpha - 1$ for integral α and $l = [\alpha]$ for nonintegral α; for the classes V_β always $l = 0$. In addition, let us agree to denote $\rho = \alpha/\nu$.

Theorem 1. For any function $u \in W_p^\alpha(Q_\nu)$ $(\alpha p > \nu)$ and any n there exists a function $f \in P(n,l)$ such that

$$\sup_{x\in Q_\nu} |u(x) - f(x)| \le Cn^{-\rho} N(u;\alpha,p,Q_\nu), \qquad C = C(\alpha,p,\nu). \tag{2}$$

For any function $u \in V_\beta$ and any n there exists a function $f \in P(n;0)$ such that

$$\sup_{x\in[0,1]} |u(x) - f(x)| \le Cn^{-1/\beta}, \qquad C = C(\beta). \tag{3}$$

Let now σ be an arbitrary finite Borel measure defined on subsets of Q_ν and normalized by the condition $\sigma(Q_\nu) = 1$. The following theorem concerns the approximation of functions of the class W_p^α by piecewise-polynomial functions in the metric of the space $L_q = L_q(Q_\nu;\sigma)$, $q \ge 1$.

Theorem 2. For any n there is a partition Λ, $|\Lambda| \le n$, such that for every function $u \in W_p^\alpha(Q_\nu)$ $(\alpha p > \nu)$ there exists a function $f \in P(\Lambda;l)$ such that

$$\|u - f\|_{L_q} \le Cn^{-\gamma} N(u;\alpha,p,Q_\nu), \qquad C = C(\alpha,p,q,\nu). \tag{4}$$

Here $\gamma = \rho$ for $q \le p$ and $\gamma = \rho - p^{-1} + q^{-1}$ for $q > p$.

An analogous assertion is valid for $u \in V_\beta$, with

$$\|u - f\|_{L_q} \le Cn^{-\delta} V(u;\beta), \qquad \delta = \min(\beta^{-1}, q^{-1}), \qquad C = C(\beta,q). \tag{5}$$

The constants C in inequalities (4), (5) do not depend on the measure σ.

The assertion of Theorem 2 is well known for the case of Lebesgue measure. In that case, obviously, as Λ one may take the partition into equal cubes, and the estimates (4), (5) become exact in order.

We note that in Theorem 1 the partition Λ depends essentially on the function u. On the contrary, in Theorem 2 the partition does not depend on the function u, although it does depend on the measure σ.

2. The proof of inequalities (2) and (4) is based on a lemma which is perhaps of independent interest. Let $J(\Delta)$ be a nonnegative function of half-open ν-dimensional parallelepipeds $\Delta \subset Q_\nu$, superadditive from below, i.e. $J(\Delta' \cup \Delta'') \geq J(\Delta') + J(\Delta'')$ for any nonintersecting Δ', Δ''. Let μ be Lebesgue measure, $a > 0$ some number, and let Λ be a partition of Q_ν into cubes Δ_k, $k = 1, \dots, |\Lambda|$. Consider the following function of partitions:

$$g_a(\Lambda) = \max_{k=1,\dots,|\Lambda|} J(\Delta_k). \tag{6}$$

Lemma. For any n there is a partition Λ, $|\Lambda| \leq n$, of the cube Q_ν such that

$$g_a(\Lambda) \leq C n^{-(1+a)} J(Q_\nu), \qquad C = C(\nu, a), \tag{7}$$

where the constant C does not depend on the function J.

We note that the order of the estimate (7) cannot be improved simultaneously for all functions J.

The proof of the lemma is connected with the construction of a special sequence of partitions Λ_i, $i = 0, 1, \dots$, of the cube Q_ν. The initial one is the trivial partition Λ_0. If the partition Λ_i has already been constructed and Δ is one of the cubes of this partition for which the maximum in (6) is attained, then the transition to the partition Λ_{i+1} is carried out by dividing the cube Δ into 2^ν equal cubes. It can be shown that for the partitions Λ_i inequality (7) is satisfied.

Let us now indicate the way in which the proof, for example, of Theorem 1 is reduced to the lemma. From the embedding theorems of S. L. Sobolev it is not hard to conclude that in any cube $\Delta \subset Q_\nu$, for a function $u \in W_p^\alpha$, one can choose a polynomial $r(x)$ of degree l such that

$$\sup_{x \in \Delta} |u(x) - r(x)| \leq C |\mu(\Delta)|^{\rho - p^{-1}} N(u; \alpha, p, \Delta), \qquad C = C(\alpha, p, \nu).$$

It remains then to apply the lemma with $J(\Delta) = N^p(u; \alpha, p, \Delta)$, $a = \varrho p - 1$. In the case $\nu = 1$ the proof can be based on other, somewhat simpler considerations, which make it possible to obtain the required result also for the classes V_β.

3. Theorem 2 can be applied to obtain estimates of the singular numbers (and hence also of the eigenvalues) of integral operators. This application

is based on the use of the following assertion, essentially due to G. Weyl ([3]) (see also ([4])).

Let $\mathfrak{H}_1, \mathfrak{H}_2$ be Hilbert spaces, and let A, A_n be operators from $\mathfrak{H}_1$ into $\mathfrak{H}_2$, with $A \in \mathfrak{S}_2$, i.e. A is a Hilbert–Schmidt operator, while A_n is some operator of finite rank n. Then for the singular numbers $s_k(A)$ of the operator A the estimate

$$\sum_{k=n+1}^{\infty} s_k^2(A) \leqslant \|A - A_n\|_{\mathfrak{S}_2}^2. \tag{8}$$

Now let $\mathfrak{H}_1 = L_2(X;\sigma)$, $\mathfrak{H}_2 = L_2(Y;\tau)$, where (X, Σ_1, σ), (Y, Σ_2, τ) are two measure spaces, with $\sigma(X) = \tau(Y) = 1$, and let $A \in \mathfrak{S}_2$ be an integral operator from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ with kernel $A(x, y)$:

$$(Au)(y) = \int_X A(x,y)u(x)\, d\sigma \qquad (x \in X,\ y \in Y).$$

Theorem 3. Let $X = Q_\nu$, and for almost all $y \in Y$ let the kernel $A(x, y)$, as a function of x, belong to the class $W_p^\alpha(Q_\nu)$ $(\alpha p > \nu)$, with

$$\int_Y \|A(\cdot, y)\|_{W_p^\alpha}^2\, d\tau = M^2 < \infty.$$

Then there exists a constant C, independent of the measures σ and τ, such that

$$s_n(A) \leqslant CMn^{-(\varrho+\vartheta)}, \qquad \vartheta = \min(1/2, 1 - 1/p), \qquad C = C(\alpha, p, \nu). \tag{9}$$

If $\nu = 1$, $A(x, y) \in V_\beta$ with respect to the variable x for almost all $y \in Y$, and

$$\int_Y \|A(\cdot, y)\|_{V_\beta}^2\, d\tau = K^2 < \infty, \tag{10}$$

then, with a constant C independent of the measures σ and τ,

$$s_n(A) \leqslant CKn^{-\chi}, \qquad \chi = \min(1, 1/2 + 1/\beta).$$

The same is true when the roles of the variables x and y are interchanged.

Estimate (9) follows from known results (see ([5–8])) only in the case when σ is Lebesgue measure. We note, however, that in the important case $p = 2$ (and consequently also for $p < 2$) estimate (9) can be obtained without reference to Theorem 2, if Weyl’ s method is combined with the suitably developed method of M. G. Krein ([11]). Let us also point out that for $\nu = 1$ the results of Theorem 3 can be obtained by a method based on estimates of entire functions and going back to Fredholm ([9,10]). This method was used to estimate singular numbers in the authors’ work ([2]), but the results given there are less complete.

Using the lemma and inequality (8), one can also prove the following assertion.

Theorem 4. Let the kernel $A(x, y)$ satisfy condition (10) with $\beta < 2$ and, moreover, for almost all $y \in Y$, satisfy in x a Lipschitz condition $\mathrm{Lip}\,\varepsilon$, $\varepsilon > 0$, with a constant L independent of y. Then

$$s_n(A) \leqslant CK^{\beta/2}L^{1-\beta/2}n^{-\lambda}, \qquad \lambda = 1 + \varepsilon(1 - \beta/2),$$

where the constant $C = C(\beta, \varepsilon)$ is independent of the measures σ and τ.

From Theorems 3 and 4 the following corollary follows immediately.

Corollary. Under the hypotheses of Theorem 4, and also under the hypotheses of the first part of Theorem 3 with $p \leqslant 2$, the operator A is nuclear.

Theorems 3 and 4 remain valid if X is a smooth compact ν-dimensional manifold without boundary or with boundary. For noncompact manifolds one can often use the results of these theorems after a suitable change of coordinates.

4. An analysis of the method of approximation used to prove Theorem 1 makes it possible to estimate, for the unit ball $S^{\alpha}_{p,\nu}$ of the space $W^{\alpha}_p(Q_\nu)$, its ε-entropy ([12]) in the space $C(Q_\nu)$.

Theorem 5. *The entropy $H_\varepsilon(S^{\alpha}_{p,\nu})$ of the compact set $S^{\alpha}_{p,\nu}$ $(\alpha p > \nu)$ in the space $C(Q_\nu)$ satisfies the inequality* *

$$H_\varepsilon(S^{\alpha}_{p,\nu}) \le C(1/\varepsilon)^{\rho^{-1}}. \tag{11}$$

The order ρ^{-1} in estimate (11) is sharp. Indeed, a lower estimate (of the same order) is valid, as is known ([12]), even for the ε-entropy in $C(Q_\nu)$ of the unit ball of the space $C^{\alpha}(Q_\nu)$ (i.e. the space of functions having smoothness of order α in the classical sense). Thus, for $\alpha p > \nu$ the order of the ε-entropy in $C(Q_\nu)$ of the unit balls of the spaces $W^{\alpha}_p(Q_\nu)$ and $C^{\alpha}(Q_\nu)$ is the same. Let us also note that for the unit ball S_β of the space V_β a result similar to Theorem 5 is impossible, since S_β is not a compact set in L_∞.

Finally, let us note the following assertion, which is a simple interpretation of Theorem 2 in terms of A. N. Kolmogorov' s n-widths ([13]).

Theorem 6. *In the metric $L_q(Q_\nu; \sigma)$, the n-widths d_n of the compact sets $S^{\alpha}_{p,\gamma}$ $(\alpha p > \nu)$ and S_β satisfy the inequalities*

$$d_n(S^{\alpha}_{p,\nu}) \le Cn^{-\gamma}, \qquad d_n(S_\beta) \le Cn^{-\delta}, \tag{12}$$

where the exponents γ and δ are the same as in Theorem 2. The constants C in inequalities (12) do not depend on the measure σ.

5. The results stated above remain valid if, instead of the classes W^{α}_p, one considers the classes H^{α}_p of S. M. Nikol' skii ([14]) or the classes B^{α}_p of O. V. Besov ([14]). Some of the results admit extension to abstract functions.

Leningrad State University
named after A. A. Zhdanov

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* V. M. Tikhomirov kindly informed us of another proof of inequality (11), valid for $\nu = 1$ and integer l.

Note: Figure translations are in progress. See original paper for figures.