GENERALIZED APPELL FUNCTIONS WITH FUNCTIONAL COEFFICIENTS

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Abstract

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GENERALIZED APPELL FUNCTIONS WITH FUNCTIONAL COEFFICIENTS

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1. Appell functions—analytic functions single-valued on a cut Riemann surface which, upon passing across the canonical cuts l_ν, undergo the linear substitution

$$\Phi^+ = m_\nu \Phi^- + p_\nu \tag{1}$$

with constant coefficients m_ν and p_ν—have been studied mainly in works ([3]–[6]). In the present article, by methods of the theory of boundary-value problems, questions of the existence of generalized analytic Appell functions with functional coefficients are solved.

Let on a normal fundamental polygon R of genus ρ of some group S of fractional-linear transformations, which is a plane topological model of a closed Riemann surface of genus ρ, there be given a system of differential equations

$$\frac{\partial U(z)}{\partial \bar{z}} = A(z)\overline{U}(z), \tag{2}$$

in which $A(z)$ is covariant with respect to $\bar{z}$ and continuous in z, with the exception of a finite number of points and discontinuity lines Γ of the first kind.

It is required to find meromorphic solutions of equation (2) having, at the prescribed points $z_1, z_2, \dots, z_m$, poles of orders respectively not exceeding $\lambda_1, \lambda_2, \dots, \lambda_m$, under the boundary condition

$$U^+[\alpha(t)] = G(t)U^+(t) + g(t), \qquad G(t),\ g(t) \in H,\ G(t) \neq 0,\ t \in L, \tag{3}$$

where L is the boundary of R. The function $\alpha(t)$ on each side of the polygon R coincides with the fractional-linear transformation that carries this side into its equivalent one; in the general case it has discontinuities of the first kind at the vertices of R and possesses the property $\alpha[\alpha(t)] = t$.

If problem (3) has a solution, then either the quotient

$$\frac{g(t) + g[\alpha(t)]G(t)}{1 - G(t)G[\alpha(t)]} \tag{4}$$

is the contour value of some meromorphic function $U(z)$ in R, or

$$G(t)G[\alpha(t)] = 1, \qquad g(t) + g[\alpha(t)]G(t) = 0. \tag{5}$$

In the first case the problem is solved directly. Consider the case when conditions (5) are fulfilled.

If $A(z) \equiv 0$ and $G(t) \equiv \text{const}$, $g(t) \equiv \text{const}$ on each side of the polygon R, then from (3) we obtain a problem whose solutions will be Appell functions ([1]). The author has solved ([8,9]) two particular cases of the problem posed.

2. In ([1]) it is shown that every regular solution of equation (2), continuous on L, can be represented in the form

$$U(z) = \Phi(z) + \iint_R \Gamma_1(z,t)\Phi(t)\,dR + \iint_R \Gamma_2(z,t)\overline{\Phi(t)}\,dR, \tag{6}$$

where $\Phi(z)$ is a function holomorphic in R; $\Gamma_1(z,t)$, $\Gamma_2(z,t)$ are the resolvents of the integral equation

$$U(z) + \frac{1}{\pi}\iint_R \frac{A(t)\overline{U(t)}}{t-z}\,dR = \Phi(z).$$

Substitute into (6) the piecewise holomorphic function

$$\Phi(z) = \frac{1}{2\pi i}\int_L \left\{ \frac{1}{\tau - t} + \sum_k \frac{1}{\tau - S_k(z)} - \frac{1}{\tau - z_0} - \sum_k \frac{1}{\tau - S_k(z)} \right\} \varphi(\tau)\,d\tau \equiv$$

$$\equiv \frac{1}{2\pi i}\int_L A(z,\tau)\varphi(\tau)\,d\tau,$$

where $z_0 \in R$, $\varphi(t) \in H$, and $S_k(z)$ are such fractional-linear transformations for which the polygons $S_k(R)$ have with R either a common side or a common vertex. Then the integral

$$U(z) = \frac{1}{2\pi i}\int_L \Omega_1^*(z,\tau)\varphi(\tau)\,d\tau - \frac{1}{2\pi i}\int_L \Omega_2^*(z,\tau)\overline{\varphi(\tau)}\,d\overline{\tau}$$

will be a definite analogue of an integral of Cauchy type on R.

The functions

$$\Omega_1^*(z,t) = A(z,t) + \iint_R \Gamma_1(z,\sigma)A(\sigma,t)\,dR,$$

$$\Omega_2^*(z,t) = \iint_R \Gamma_2(z,\sigma)\overline{A(\sigma,t)}\,dR$$

will be called ([1]) kernels of the differential equation (2) for the domain R. It is easy to verify that Ω_1^* and Ω_2^* possess all the properties of the kernels Ω_1, Ω_2 introduced by I. N. Vekua (([1]), § 5).

The analogues needed by us of the Sokhotski formulas

$$U^+(t) = \tfrac{1}{2}\varphi(t) - \tfrac{1}{2}\varphi[a(t)] + U(t),$$

$$U^+(t) = (1-\gamma_k/2\pi)[\varphi(t) - \varphi(a(t))] + U(t) \tag{7}$$

(the second for corner points) can be obtained by the usual device, taking into account the orientation of the sides and the relation $a[a(t)] \equiv t$.

3. The regular solution $U_1(z)$, vanishing at the point $z = z_0$, of the "jump" problem is sought in the form

$$U_1(z) = \frac{1}{2\pi i}\int_L \Omega_1^*(z,\tau)\varphi(\tau)\,d\tau - \frac{1}{2\pi i}\int_L \Omega_2^*(z,\tau)\overline{\varphi(\tau)}\,d\overline{\tau}$$

under the condition that

$$\varphi(t) + \varphi[a(t)] = 0. \tag{8}$$

In this way, by means of (7) and (8), the boundary condition of the problem is reduced to the integral equation

$$\varphi(t) + \int_L R_1(t,\tau)\varphi(\tau)\,d\tau + \int_L R_2(t,\tau)\overline{\varphi(\tau)}\,d\overline{\tau} = -\frac{1}{2}g(t), \tag{9}$$

in which

$$R_1(t,\tau) = \frac{1}{4\pi i}\{\Omega_1^*(t,\tau) - \Omega_1^*[a(t),a(\tau)]\alpha'(\tau)\},$$

$$R_2(t,\tau) = \frac{1}{4\pi i}\{\Omega_2^*(t,\tau) - \Omega_2^*[a(t),a(\tau)]\overline{\alpha'(\tau)}\}$$

turn out to be Fredholm kernels.

The second formula (7) leads to an equation of the same type. It can be shown that, when condition (5) is satisfied, equation (9) has a unique solution.

The meromorphic solution $U(z)$ with the singularities prescribed in the conditions of the problem is found from the equality $U(z) - U_1(z) = U_\lambda(z)$, taking into account that U_λ is a generalized automorphic function—a generalized automorphic polynomial of degree $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_m$.

Such a generalized polynomial, following I. N. Vekua ([1]), § 5, is not difficult to construct if one starts from formula (9)

$$W(z) = \psi(z)\left\{1 + \iint_R \Gamma_1^*(z,t)\,dR + \iint_R \Gamma_2^*(z,t)\,dR\right\},$$

which assigns to each automorphic function $\psi(z)$ a generalized automorphic function $W(z)$. Here $\Gamma_1^*(z,t)$, $\Gamma_2^*(z,t)$ are the resolvents of the equation

$$\frac{\partial V}{\partial z} = A\,\frac{\overline{\psi}}{\psi}\,\overline{V}.$$

In this way, for 1, i, and the elementary Weierstrass automorphic functions ([7]) $H_k(z,z_j)$, $iH_k(z,z_j)$, $k = 0,1,\dots,\lambda_j - 1$, $j = 1,2,\dots,m$, we construct on the fundamental polygon R generalized analogues of the degrees $U_1(z)$, $U_1^i(z)$, $U_{H_k}(z,z_j)$, $U_{H_k}^i(z,z_j)$, and

$$U_\lambda(z) = \alpha_1 U_1(z) + \alpha_1^i U_1^i(z) + \sum_{j=1}^{m}\sum_{k=0}^{\lambda_j - 1}\left[\alpha_{j,k} U_{H_k}(z,z_j) + \alpha_{j,k}^i U_{H_k}^i(z,z_j)\right].$$

Obviously, the constants α_1, α_1^i, $\alpha_{j,k}$, $\alpha_{j,k}^i$ here must be chosen so that U_λ has no poles at the points $z = a_\sigma$ ([7]).

4. To obtain the solution of the homogeneous problem, we construct, according to the scheme ([7,8]), the canonical function

$$X(z) = e^{\Gamma(z)} \prod_{k=1}^{q}\prod_{j=1}^{x_k} E(z, Q_{k,j}; t_k) \prod_{k=q+1}^{4p}\prod_{j=1}^{|x_k|} E(z, t_k; Q_{k,j}),$$

in which

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \Omega_1^*(z,\tau)\varphi(\tau)\,d\tau - \frac{1}{2\pi i}\int_L \Omega_2^*(z,\tau)\overline{\varphi(\tau)}\,d\overline{\tau},$$

and $\varphi(\tau)$ is the solution of integral equation (9) with right-hand side equal to $-\frac{1}{2}\ln G(t)$.

The function $X(z)$ has, at the points $Q_1, Q_2, \dots, Q_p$ distinct among $Q_{k,j}$, $k = 1, 2, \dots, q$; $j = 1, 2, \dots, x_k$, zeros of multiplicities $\mu_1, \mu_2, \dots, \mu_p$, and at the points $Q_1^*, Q_2^*, \dots, Q_R^*$ (distinct among $Q_{k,j}$, $k = q+1, \dots, 4p$; $j = 1, 2, \dots, |x_k|$) poles of orders $\nu_1, \nu_2, \dots, \nu_l$, respectively.

The boundary conditions of the homogeneous problem can now be written as

$$U^+[\alpha(t)]/X^+[\alpha(t)] = U^+(t)/X^+(t),$$

again reducing the determination of the meromorphic solution $U(z)$ to the construction of an analogue of the generalized polynomial $U_{\lambda+\mu}(z) = U(z)/X(z)$ from its prescribed singularities.

Therefore

$$\begin{aligned} U(z) = X(z)\Bigg\{ & \alpha_1 U_1(z) + \alpha_1^i U_1^i(z) + \\ & + \sum_{j=1}^{m} \sum_{k=0}^{\lambda_j - 1} \left[\alpha_{j,k} U_{H_k,X}(z, z_j) + \alpha_{j,k}^i U_{H_k,X}^i(z, z_j)\right] + \\ & + \sum_{j=1}^{p} \sum_{k=0}^{\mu_j - 1} \left[\beta_{j,k} U_{H_k,X}(z, Q_j) + \beta_{j,k}^i U_{H_k,X}^i(z, Q_j)\right] \Bigg\}. \end{aligned} \tag{10}$$

The undetermined coefficients $U_{\lambda+\mu}$ are chosen here so that the function $U(z)$ has no "extraneous" poles at the points $z = a_\sigma$, and $U_{\lambda+\mu}$ vanishes at the points $Q_1^*, Q_2^*, \dots, Q_l^*$ a prescribed number of times.

5. If the boundary condition of the nonhomogeneous problem (3) is written in the form

$$U^+[\alpha(t)]/X^+[\alpha(t)] = U^+(t)/X^+(t) + g(t)/X^+[\alpha(t)],$$

then, as before, we obtain

$$U(z) = X(z)\left\{\frac{1}{2\pi i}\int_L \Omega_1^*(z,\tau)\varphi(\tau)\,d\tau - \frac{1}{2\pi i}\int_L \Omega_2^*(z,\tau)\overline{\varphi(\tau)}\,\overline{d\tau} + U_{\lambda+\mu}(z)\right\},$$

where $\varphi(t)$ is the solution of the integral equation (9) with right-hand side equal to $-\frac{1}{2}g(t)/X^+[\alpha(t)]$.

Eliminating the "extraneous" poles of the polynomial $U_{\lambda+\mu}$ at the points $z = a_\sigma$, $z = Q_\beta$, $\beta = 1, 2, \dots, l$, and ensuring the regularity of $U(z)$ at the poles Q^*_α, $\alpha = 1, 2, \dots, p$, of the function $X(z)$ by choosing the coefficients, we obtain the system

$$\sum_{j=1}^{m} \sum_{k=0}^{\lambda_j - 1} c_{j,k} h^j_{\sigma,k} + \sum_{j=1}^{p} \sum_{k=0}^{\mu_j - 1} B_{j,k} b^j_{\sigma,k} = 0,$$

$$\left[\frac{1}{2\pi i} \int_L A(Q^*_\eta, \tau) \varphi(\tau) \, d\tau + c + \sum_{j=1}^{m} \sum_{k=0}^{\lambda_j - 1} c_{j,k} H_k(Q^*_\eta, z_j) + \sum_{j=1}^{p} \sum_{k=0}^{\mu_j - 1} B_{j,k} H_k(Q^*_\eta, z_j) \right]^{(\xi)} = 0,$$

$$\xi = 0, 1, \dots, \nu_\eta - 1; \qquad \eta = 1, 2, \dots, l,$$

in which c, $c_{j,k}$, $B_{j,k}$ are fully determined complex constants expressed in terms of the coefficients of $U_{\lambda+\mu}$.

The obtained system, consisting of $\nu + \rho$ equations $(\nu = \nu_1 + \nu_2 + \dots + \nu_l)$, contains $\lambda + \mu + 1$ $(\mu = \mu_1 + \mu_2 + \dots + \mu_p)$ unknowns. Investigation of this system leads to the following conclusion.

If $\chi = \mu - \nu$ is the index of the problem, then for $\chi + \lambda > \rho - 1$ problem (3) has $\lambda + \chi - \rho + 1$ linearly independent solutions; for $\chi + \lambda = \rho - 1$, it has a unique solution, determined by formula (10); whereas when $\lambda + \chi < \rho - 1$, in order for the problem to be solvable in the general case it is necessary to impose additional conditions on the function $g(t)$.

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Note: Figure translations are in progress. See original paper for figures.