ON THE BEST QUADRATURE FORMULAS

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Abstract

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ON THE BEST QUADRATURE FORMULAS FOR CERTAIN CLASSES OF FUNCTIONS

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In the present note we shall study problems whose general formulation is given in the works of S. M. Nikol' skii ([1,2]).

1. Denote by $W^{(r)}_{L_q}(M;0,1)$, where r is a natural number, $1 \leq q \leq \infty$, the class of functions having on the interval $[0,1]$ an absolutely continuous derivative of order $r-1$ and, for $1 \leq q < \infty$, a derivative $f^{(r)}(x)$ of order r satisfying the inequality

$$\left(\int_0^1 |f^{(r)}(x)|^q \, dx\right)^{1/q} \leq M,$$

and for $q = \infty$ a piecewise continuous derivative $f^{(r)}(x)$ satisfying on the given interval the inequality

$$|f^{(r)}(x)| \leq M.$$

S. M. Nikol' skii ([1,2]) introduced into consideration quadrature formulas

$$\int_0^1 f \, dx \simeq \frac{1}{r!} \sum_{k=0}^{m-1} \sum_{l=0}^{r-2} \lambda_k^{(l)} (r-l-1)! \, f^{(l)}(x_k) = L(f), \tag{1}$$

defined by given nodes x_k $(0 \leq x_0 < x_1 < \cdots < x_{m-1} \leq 1)$ and coefficients $\lambda_k^{(l)}$.

Suppose that the quadrature formula (1) is exact for every polynomial of degree $r-1$. For such a quadrature formula it is not difficult to find (see ([2])) the exact estimate

$$\sup_{f\in W^{(r)}_{L_q}(M;0,1)} \left|\int_0^1 f\,dx - L(f)\right| = M\|F_r(t)\|_{L_p} \quad \left(1 \le q \le \infty,\ \frac{1}{p}+\frac{1}{q}=1\right), \quad (2)$$

where

$$F_r(t) = \frac{1}{r!}\left[(1-t)^r - \sum_{k=0}^{m-1}\sum_{l=0}^{r-2}\lambda_k^{(l)}K_{r-l}(x_k-t)\right], \quad (3)$$

$$K_s(u) = \begin{cases} u^{s-1}, & \text{for } u \ge 0, \\ 0, & \text{for } u < 0, \end{cases}$$

$$\|F_r(t)\|_{L_p} = \begin{cases} \left(\int_0^1 |F_r(t)|^p\,dt\right)^{1/p}, & \text{for } 1 \le p < \infty, \\ \max\limits_{0\le t\le 1} |F_r(t)|, & \text{for } p = \infty. \end{cases}$$

We shall seek, among all possible quadrature formulas (1) exact for every polynomial of degree $r-1$, the best one for the class $W^{(r)}_{L_q}(M;0,1)$, i.e. one for which the quantity (2) has the least

value. We shall solve the problem posed under the assumption that r is an even number.

Let us note that the quadrature formula (1) is exact for any polynomial of degree $r-1$ if and only if, for all t, the condition

$$L\left[(x-t)^{r-1}\right] = \int_0^1 (x-t)^{r-1}\,dx,$$

i.e.

$$\sum_{k=0}^{m-1}\sum_{l=0}^{r-2}\lambda_k^{(l)}(x_k-t)^{r-l-1} = (1-t)^r - t^r. \quad (4)$$

is satisfied.

Taking this into account, we obtain that the solution of our problem consists in finding, for fixed m and r, the least value of the quantity (2), if the numbers x_k are varied ($0 \le x_0 < x_1 < \cdots < x_{m-1} \le 1$) and the $\lambda_k^{(l)}$, which satisfy condition (4). The solution of this problem is based on the use of properties of the polynomial $R_{r,p}(x)$ of degree r, with leading coefficient equal to one, which deviates least from zero on the interval $[-1,1]$ in the metric L_p.

Theorem 1. Among all possible quadrature formulas (1), exact for any polynomial of degree $r-1$ (m, r fixed natural numbers, with r an even number), the best for the class $W^{(r)}_{L_q}(M;0,1)$ $(1 \le q \le \infty)$ is the unique formula

$$\int_0^1 f\,dx \approx L_*(f)$$

with nodes x_k^* and coefficients $\lambda^{(l)}_{k^*}$, expressed by the equalities

$$x_k^* = \left(2k + \sqrt[r]{R_{r,p}(1)}\right) h_* \qquad (k = 0, 1, \dots, m-1),$$

$$\lambda^{(l)}_{0*} = \frac{h_*^{l+1}}{(r-l-1)!} \left\{ (-1)^{r-l} \frac{r!}{(l+1)!} \left[R_{r,p}(1)\right]^{(l+1)/r} + R^{(r-l-1)}_{r,p}(1) \right\}$$

$$(l = 0, 1, \dots, r-2),$$

$$\lambda^{(2i+1)}_{k*} = 0 \qquad (i = 0, 1, \dots, (r-4)/2;\ k = 1, \dots, m-2),$$

$$\lambda^{(2i)}_{k*} = \frac{2h_*^{2i+1}}{(r-2i-1)!} R^{(r-2i-1)}_{r,p}(1) \qquad (i = 0, 1, \dots, (r-2)/2;\ k = 1, \dots, m-2),$$

$$\lambda^{(l)}_{m-1,*} = \frac{h_*^{l+1}}{(r-l-1)!} \left\{ \frac{r!}{(l+1)!} \left[R_{r,p}(1)\right]^{(l+1)/r} - R^{(r-l-1)}_{r,p}(-1) \right\}$$

$$(l = 0, 1, \dots, r-2),$$

where

$$h_* = \frac{1}{2\left[m - 1 + \sqrt[r]{R_{r,p}(1)}\right]}.$$

For this formula the sharp estimate holds

$$\sup_{f \in W^{(r)}_{L_q}(M;0,1)} \left| \int_0^1 f\,dx - L_*(f) \right| = M \frac{h_*^r R_{r,p}(1)}{r! \sqrt[r]{rp+1}}.$$

2. Denote by $W_{0L_q}^{(r)}(M;0,1)$ the class of functions consisting of all functions of the class $W_{L_q}^{(r)}(M;0,1)$ satisfying the additional condition $f(0) = f'(0) = \dots = f^{(r-1)}(0) = 0$. For any quadrature formula (1), regardless of whether it is exact or inexact for all

polynomials of degree $r-1$, it is not difficult to find (see (2)) the exact estimate

$$\sup_{f \in W_0^{(r)} L_q(M;0,1)} \left| \int_0^1 f\,dx - L(f) \right| = M \|F_r(t)\|_{L_p} \quad \left(1 \le q \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1\right), \tag{5}$$

where $F_r(t)$ is still defined by equality (3).

We shall seek, among all possible quadrature formulas (1), the best one for the class $W_0^{(r)} L_q(M;\,0,1)$. Our problem, obviously, consists in finding the smallest value of the quantity (5), if one varies the numbers x_k $(0 \le x_0 < x_1 < \dots < x_{m-1} \le 1)$ and $\lambda_k^{(l)}$, where m and r are fixed natural numbers. We also solve this problem under the assumption that r is an even number.

Theorem 2. *Among all possible quadrature formulas (1), where m, r are fixed natural numbers, with r an even number, the best for the class $W_0^{(r)} L_q(M;\,0,1)$ $(1 \le q \le \infty)$ is the unique formula*

$$\int_0^1 f\,dx \simeq L_*(f)$$

with nodes x_{k*} and coefficients $\lambda_{k*}^{(l)}$, expressed by means of the equalities

$$x_{k*} = 2(k+1)h_* \quad (k = 0, 1, \dots, m-1),$$

$$\lambda_{k*}^{(2i+1)} = 0 \quad (i = 0, 1, \dots, (r-4)/2;\ k = 0, 1, \dots, m-2),$$

$$\lambda_{k*}^{(2i)} = \frac{2h_*^{2i+1}}{(r-2i-1)!} R_{r,p}^{(r-2i-1)}(1)$$

$$(i = 0, 1, \dots, (r-2)/2; \quad k = 0, 1, \dots, m-2),$$

$$\lambda_{m-1,*}^{(l)} = \frac{h_*^{l+1}}{(r-l-1)!} \left\{ \frac{r!}{(l+1)!} [R_{r,p}(1)]^{(l+1)/r} - R_{r,p}^{(r-l-1)}(-1) \right\}$$

$$(l = 0, 1, \dots, r-2),$$

where

$$h_* = \frac{1}{2m + \sqrt[r]{R_{r,p}(1)}}.$$

For this formula the following exact estimate holds:

$$\sup_{f \in W_0^{(r)} L_q(M;0,1)} \left| \int_0^1 f\,dx - L_*(f) \right| = M\,\frac{h_*^r R_{r,p}(1)}{r!\,\sqrt[p]{rp+1}}.$$

Remark. In the case $r = 2$, $1 \le q \le \infty$, Theorem 1 was proved by T. A. Shaidaeva ([3]). For arbitrary even r, $q = 1$, $q = 2$, and $q = \infty$, Theorem 1 was proved by I. I. Ibragimov and R. M. Aliev; they also proved Theorem 2 for arbitrary even r, $q = 2$ (this was reported by R. M. Aliev at the seminar on constructive function theory at the V. I. Lenin Belarusian State University). We note that in the case $r = 2$, $q = 2$, Theorem 2 had earlier also been proved by G. Ya. Doronin ([4]). For arbitrary even r and $q = \infty$, Theorem 2 was proved by S. M. Nikol' skii ([1,2]), and for $q = 1$ by A. I. Kiselev (see ([2]), pp. 107-109).

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Note: Figure translations are in progress. See original paper for figures.