ASYMPTOTICS OF GREEN' S FUNCTIONS

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ASYMPTOTICS OF GREEN' S FUNCTIONS

IN THE VISCOUS HYDRODYNAMIC APPROXIMATION

FOR THE STATISTICAL MECHANICS OF QUANTUM SYSTEMS

(Presented by Academician N. N. Bogolyubov, 27 X 1965)

In note ([1]) we proposed a method for constructing asymptotic expressions for Fourier components of "classical" Green' s functions ([2]) in the viscous hydrodynamic approximation. The aim of the present article is to extend the indicated method to the case of the statistical mechanics of quantum systems.

The principles for constructing the hydrodynamic approximation for quantum many-body systems were developed in ([3,4]) on the basis of a study of the general quantum equations of motion. Some questions concerning the derivation of the equations of two-fluid hydrodynamics were also considered in a recent work ([5]) in connection with the study of the simplest possibilities for decoupling the chain of equations for Green' s functions.

In the present work we shall proceed from the quantum-mechanical analogue of the Boltzmann equation ([6,7])

$$\frac{\partial f_1}{\partial t} + \sum_{(1\leqslant\alpha\leqslant 3)} \frac{p_1^\alpha}{m}\frac{\partial f_1}{\partial r_\alpha} = \frac{\pi}{(2\pi)^6}\int dp_2\, dp_1'\, dp_2'\, \left[\Phi\left(|\mathbf{p}_1-\mathbf{p}_1'|\right)+\Phi\left(|\mathbf{p}_1-\mathbf{p}_2'|\right)\right]^2 \times$$

$$\times\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')\delta\left(\frac{p_1^2}{2m}+\frac{p_2^2}{2m}-\frac{{p_1'}^2}{2m}-\frac{{p_2'}^2}{2m}\right)\times$$

$$\times\{f_1'f_2'(1+\Omega f_1)(1+\Omega f_2)-f_1f_2(1+\Omega f_1')(1+\Omega f_2')\}. \quad (1)$$

Here $f_1 = f_1(\mathbf{r}_1,\mathbf{p}_1,t)$; $f_1' = f_1'(\mathbf{r}_1,\mathbf{p}_1',t)$, etc.; $\Omega = -1$ in the case of Fermi statistics; $\Omega = +1$ in the case of Bose statistics; for $\Omega = 0$, equation (1) includes the Boltzmann equation of classical statistics. The expression in square brackets corresponds to the choice of the scattering-process cross section in the Born approximation.

The functions $f(\mathbf{p}, \mathbf{r})$ are related to the quantum correlation functions $F(q, q')$, introduced in the monograph of N. N. Bogolyubov ([8]), by the relation

$$f(\mathbf{p}, \mathbf{r}) = \frac{1}{(2\pi\hbar)^3} \int F(\mathbf{r} + \vec{\xi}/2; \mathbf{r} - \vec{\xi}/2) e^{i\mathbf{p}\vec{\xi}/\hbar} \, d\vec{\xi}.$$

Applying the procedure developed by us in ([2,9]) and following the notation of ([1]), the initial equation (1) may be written in terms of Green' s functions as

$$-E\Lambda_{E\vec{v}}(\mathbf{p}) + \sum_{\alpha} \frac{p_\alpha v_\alpha}{m} \Lambda_{E\vec{v}}(\mathbf{p}) - E\Lambda_{0\vec{v}}(\mathbf{p}) = L(\Lambda_{E\vec{v}}), \tag{2}$$

where

$$\Lambda_{E\vec{v}}(\mathbf{p}) = f^0(\mathbf{p}) \int G_{E\vec{v}}(\mathbf{p}, \mathbf{p}') F(\mathbf{p}') \, d\mathbf{p}',$$

$F(\mathbf{p}')$ is a regular function, whose specification will be made below;

$$G_{E\mathbf{v}}(\mathbf{p}, \mathbf{p}') = G'_{E\mathbf{v}}(\mathbf{p}, \mathbf{p}') - G'_{0\mathbf{v}}(\mathbf{p}, \mathbf{p}');$$

$f^0(\mathbf{p})$ is the equilibrium distribution function; $f^0(\mathbf{p}) = 1/(e^{p^2/2m\theta - \mu/\theta} - \Omega)$;
L is the collision operator corresponding to the integral term on the right-hand side of (1).

The study of the obtained equation (2) by the method of successive approximations (see (1)) presents no difficulty.

Here we shall consider the situation corresponding to the hydrodynamic approximation, i.e., the case of small E and $\mathbf{v}$ (“slow” processes). Just as in the classical case, from equation (2) one can directly obtain 5 equations of the type of hydrodynamic equations:

$$\begin{gathered} -E\rho^*_{E\mathbf{v}} + \rho_0 \sum_{\alpha} v_\alpha u^*_\alpha(E; \mathbf{v}) = a, \\ -E\rho_0 u^*_i(E; \mathbf{v}) + \sum_{\alpha} v_\alpha \hat{P}_{i\alpha}(E; \mathbf{v}) = b_i \quad (i = 1, 2, 3), \\ -E\hat{\mathcal{E}}_{E\mathbf{v}} + \sum_{\alpha} v_\alpha \hat{q}_\alpha(E; \mathbf{v}) = d, \end{gathered} \tag{3}$$

in which the following notation has been introduced:

$$\frac{g}{(2\pi)^3} m \int \Lambda_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p} = \rho^*_{E\mathbf{v}}, \qquad g = (2s+1), \quad s \text{ is the spin};$$

$$\frac{g}{(2\pi)^3} \int p_\alpha \Lambda_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p} = \rho_0 u^*_\alpha(E;\mathbf{v}); \qquad \rho_0 = m\frac{N}{V}; \tag{4}$$

$$\frac{g}{(2\pi)^3} \int \frac{p^2}{2m} \Lambda_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p} = \hat{\mathcal{E}}_{E\mathbf{v}};$$

$$\hat{P}_{i\alpha}(E;\mathbf{v}) = \frac{g}{(2\pi)^3} \int \frac{p_\alpha p_i}{m} \Lambda_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p};$$

$$\hat{q}_\alpha(E;\mathbf{v}) = \frac{g}{(2\pi)^3} \int \frac{p^2}{2m}\frac{p_\alpha}{m} \Lambda_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p};$$

$$a = Em\frac{g}{(2\pi)^3} \int \Lambda_{0\mathbf{v}}(\mathbf{p})\, d\mathbf{p};$$

$$b_i = E\frac{g}{(2\pi)^3} \int p_i \Lambda_{0\mathbf{v}}(\mathbf{p})\, d\mathbf{p};$$

$$d = E\frac{g}{(2\pi)^3} \int \frac{p^2}{2m} \Lambda_{0\mathbf{v}}(\mathbf{p})\, d\mathbf{p}.$$

Next, following (1), we shall use the ideas of Hilbert and Chapman–Enskog. Put

$$\Lambda_{E\mathbf{v}}(\mathbf{p}) = \Lambda^{(0)}_{E\mathbf{v}}(\mathbf{p}) + \Lambda^{(1)}_{E\mathbf{v}}(\mathbf{p})$$

and require that the introduced quantities $\rho^*_{E\mathbf{v}}$, $u^*_\alpha(E;\mathbf{v})$, $\hat{\mathcal{E}}_{E\mathbf{v}}$ be determined only with the aid of the function $\Lambda^{(0)}_{E\mathbf{v}}(\mathbf{p})$, namely

$$\int \Lambda^{(1)}_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p} = \int p_\alpha \Lambda^{(1)}_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p} = \int p^2 \Lambda^{(1)}_{E\mathbf{v}}(\mathbf{p})\, d\mathbf{p} = 0.$$

Bearing in mind that in what follows we shall need the hydrodynamic equations only in the "acoustic" approximation, we take the function $\Lambda^{(0)}_{E\mathbf{v}}(\mathbf{p})$ in the form

$$\Lambda^{(0)}_{E\mathbf{v}} = f^0(1+\Omega f^0)\left\{\frac{1}{\theta}\mu^* + \frac{1}{\theta}\sum_\alpha p_\alpha u^*_\alpha(E;\mathbf{v}) + \left(\frac{p^2}{2m} - \mu\right)\frac{1}{\theta^2}\theta^*_{E\mathbf{v}}\right\}; \tag{5}$$

μ^*, u^*_α, $\theta^*_{E\mathbf{v}}$ are, for the time being, formally introduced quantities.

Substituting (5) into the definitions (4), we obtain

$$\hat{\mathcal{E}}_{E\vec{v}} = \frac{3}{2}\left(\frac{\partial P}{\partial \theta}\right)_0 \theta^*_{E\vec{v}} + \frac{3}{2}\left(\frac{\partial P}{\partial \rho}\right)_0 \rho^*_{E\vec{v}},$$

$$\hat{q}^{(0)}_\alpha(E;\vec{v}) = \frac{5}{2}(P)_0 u^*_\alpha(E;\vec{v}), \tag{6}$$

$$\hat{P}^{(0)}_{j\alpha}(E;\vec{v}) = \delta_{j\alpha}\left\{\left(\frac{\partial P}{\partial \rho}\right)_0 \rho^*_{E\vec{v}} + \left(\frac{\partial P}{\partial \theta}\right)_0 \theta^*_{E\vec{v}}\right\};$$

in these formulas P is the ordinary thermodynamic pressure, and $(\cdots)_0$ denotes the equilibrium state.

The expressions (6) determine the equations of the zeroth approximation, corresponding to the linearized Euler equations.

Now, representing the left-hand side of (2) in terms of the equations of the zeroth approximation and taking into account the fact that we are interested in "slow" (hydrodynamic) processes, i.e., small E and $\vec{v}$, it is not difficult to see that the equation for determining the function $\Lambda^{(1)}_{E\vec{v}}(p)$, by a simple change of variables, is reduced to an equation studied in detail in the work of Uehling and Uhlenbeck ([7]). As a result we obtain expressions for $\hat{q}^{(1)}_\alpha(E;\vec{v})$ and $\hat{P}^{(1)}_{j\alpha}(E;\vec{v})$:

$$\hat{q}^{(1)}_\alpha(E;\vec{v}) = -i\chi v_\alpha \theta^*_{E\vec{v}}.$$

$$\hat{P}^{(1)}_{j\alpha}(E;\vec{v}) = -2\eta\left\{\frac{1}{2}\left[iv_\alpha u^*_j(E;\vec{v}) + iv_j u^*_\alpha(E;\vec{v})\right] - \frac{1}{3}\delta_{j\alpha} i\sum_\beta v_\beta u^*_\beta(E;\vec{v})\right\},$$

where the viscosity coefficient η and the thermal conductivity χ have the standard form.

Substituting the values of $\hat{q}_\alpha(E;\vec{v})$ and $\hat{P}_{j\alpha}(E;\vec{v})$ into equations (3), we arrive at a system of linear inhomogeneous equations in the Navier-Stokes approximation, whose solutions may be represented in the form

$$\theta^*_{E\vec{v}} = \frac{\Delta_1(E;\vec{v})}{(E^2 - c_0^2 v^2) + i\Gamma(c_0 v;\, v) + D'(E;\vec{v})};$$

$$\rho^*_{E\vec{v}} = \frac{\Delta_2(E;\vec{v})}{(E^2 - c_0^2 v^2) + i\Gamma(c_0 v;\, v) + D'(E;\vec{v})}; \tag{7}$$

$$\xi = \frac{\Delta_3(E;\vec{v})}{(E^2 - c_0^2 v^2) + i\Gamma(c_0 v;\, v) + D'(E;\vec{v})},$$

where

$$c_0 = \left[\frac{5}{3}\left(\frac{P}{\rho}\right)_0\right]^{1/2}$$

is the speed of sound, and the damping coefficients are expressed in a natural way through the kinetic coefficients of viscosity η and thermal conductivity χ.

Using now our notation and putting, for example, $F(p') = 1;\ F(p') = p'_\alpha$, we see that the obtained solutions determine the asymptotic expressions (since our reasoning is valid in the case $E \ll 1/T,\ v \ll 1/\lambda$, λ being the mean free path and T the relaxation time) for the entire set of Green functions of the type $\langle n_i(r); n(r')\rangle_E$ (density—density), $\langle n(r); j_\alpha(r')\rangle_E$ (density—flux), $\langle j_\alpha(r); n(r')\rangle_E$ (flux—

density), $\langle\langle j_\alpha(\mathbf{r}); j_\alpha(\mathbf{r}')\rangle\rangle$ (current—current), etc., whose pole structure is completely determined by the denominators (7). The corresponding correlation functions can be constructed directly on the basis of the general method developed earlier [1].

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References

Note: Figure translations are in progress. See original paper for figures.

[1] N. N. Bogoliubov, Jr., B. I. Sadovnikov, ZhETF, **43**, 677 (1962).