

# ASYMPTOTIC CALCULATION OF CORRELATION FUNCTIONS

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**Abstract**

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*MATHEMATICAL PHYSICS*

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## ASYMPTOTIC CALCULATION OF CORRELATION FUNCTIONS

*(Presented by Academician N. N. Bogolyubov, January 8, 1966)*

In <sup>(1)</sup>, the asymptotic form (for  $k \ll 1/L$  and  $\omega \ll 1/T$ ,  $L$  and  $T$  being the characteristic length and relaxation time) of the basic correlation functions was obtained by comparing the solutions of the ordinary linearized hydrodynamic equations and the expressions that follow directly from the theorem on the variation of the mean (see, for example, <sup>(2)</sup>). In this approach, the chosen variation of the Hamiltonian has a complicated form and does not admit a simple physical interpretation. It seems more natural to us, following N. N. Bogolyubov <sup>(3,4)</sup>, to take the Hamiltonian of the system in the second-quantization representation in the form

$$\begin{aligned}
 H = & \frac{1}{2m} \int \nabla \Psi^+(t, \mathbf{r}) \nabla \Psi(t, \mathbf{r}) d\mathbf{r} + \\
 & + \frac{1}{2} \int \Phi(|\mathbf{r} - \mathbf{r}'|) \Psi^+(t, \mathbf{r}) \Psi^+(t, \mathbf{r}') \Psi(t, \mathbf{r}') \Psi(t, \mathbf{r}) d\mathbf{r} d\mathbf{r}' + \\
 & + \int U(t, \mathbf{r}) \Psi^+(t, \mathbf{r}) \Psi(t, \mathbf{r}) d\mathbf{r} - \frac{1}{m} \int \mathbf{j}(t, \mathbf{r}) \mathbf{A}(t, \mathbf{r}) d\mathbf{r} - \\
 & - \frac{1}{2m} \int \Psi^+(t, \mathbf{r}) A^2(t, \mathbf{r}) \Psi(t, \mathbf{r}) d\mathbf{r},
 \end{aligned} \tag{1}$$

where the current-density operator

$$\mathbf{j}(t, \mathbf{r}) = \frac{i}{2} [\nabla \Psi^+(t, \mathbf{r}) \Psi(t, \mathbf{r}) - \Psi^+(t, \mathbf{r}) \nabla \Psi(t, \mathbf{r})] - \Psi^+(t, \mathbf{r}) \Psi(t, \mathbf{r}) \mathbf{A}(t, \mathbf{r}),$$

and  $U(t, \mathbf{r})$  and  $\mathbf{A}(t, \mathbf{r})$  are the potential and vortex components of the external field, satisfying the condition  $\text{div } \mathbf{A} = 0$ . Then the equations of motion for the field operators have the form

$$\begin{aligned}
i \frac{\partial \Psi(t, \mathbf{r})}{\partial t} &= -\frac{\Delta}{2m} \Psi(t, \mathbf{r}) + \int \Phi(|\mathbf{r} - \mathbf{r}'|) \Psi^+(t, \mathbf{r}') \Psi(t, \mathbf{r}') d\mathbf{r}' \Psi(t, \mathbf{r}) + \\
&\quad + U(t, \mathbf{r}) \Psi(t, \mathbf{r}) + \frac{i}{m} \nabla \Psi(t, \mathbf{r}) \mathbf{A}(t, \mathbf{r}) - \frac{A^2(t, \mathbf{r})}{2m} \Psi(t, \mathbf{r}), \\
i \frac{\partial \Psi^+(t, \mathbf{r})}{\partial t} &= \frac{\Delta}{2m} \Psi^+(t, \mathbf{r}) - \Psi^+(t, \mathbf{r}) \int \Phi(|\mathbf{r} - \mathbf{r}'|) \Psi^+(t, \mathbf{r}') \Psi(t, \mathbf{r}') d\mathbf{r}' - \\
&\quad - U(t, \mathbf{r}) \Psi^+(t, \mathbf{r}') + \frac{i}{m} \nabla \Psi^+(t, \mathbf{r}) \mathbf{A}(t, \mathbf{r}) + \frac{A^2(t, \mathbf{r})}{2m} \Psi^+(t, \mathbf{r}).
\end{aligned} \tag{2}$$

With the aid of (2) one can obtain the hydrodynamic equations, which, after linearization, take the form

$$\begin{aligned}
\frac{\partial \delta n(t, \mathbf{r})}{\partial t} + n_0 \nabla \cdot \mathbf{V}(t, \mathbf{r}) &= 0, \\
mn_0 \frac{\partial \mathbf{V}(t, \mathbf{r})}{\partial t} &= -\nabla \delta P + \eta \Delta \mathbf{V}(t, \mathbf{r}) + \left( \frac{1}{3} \eta + \xi \right) \nabla \nabla \cdot \mathbf{V}(t, \mathbf{r}) - \\
&\quad - n_0 \nabla U(t, \mathbf{r}) - n_0 \frac{\partial \mathbf{A}(t, \mathbf{r})}{\partial t}, \\
n_0 \theta_0 \frac{\partial \delta S(t, \mathbf{r})}{\partial t} &= \chi \Delta \delta \theta(t, \mathbf{r}),
\end{aligned} \tag{3}$$

where

$$\delta P = \left( \frac{\partial P}{\partial n} \right)_\theta \delta n(t, \mathbf{r}) + \left( \frac{\partial P}{\partial \theta} \right)_n \delta \theta(t, \mathbf{r}), \quad \delta S = \left( \frac{\partial S}{\partial n} \right)_\theta \delta n(t, \mathbf{r}) + \left( \frac{\partial S}{\partial \theta} \right)_n \delta \theta(t, \mathbf{r}).$$

If the velocity is represented in the form

$$\mathbf{V} = \mathbf{V}_l + \mathbf{V}_\tau, \quad \text{rot } \mathbf{V}_l = 0, \quad \text{div } \mathbf{V}_\tau = 0$$

and we pass to Fourier components, we obtain

$$\begin{aligned}
-\omega \delta n(k, \omega) + n_0 \sum_\alpha k^\alpha V^\alpha(k, \omega) &= 0, \\
mn_0 \omega V_l^\alpha(k, \omega) &= k^\alpha \left[ \left( \frac{\partial P}{\partial n} \right)_0 \delta n(k, \omega) + \left( \frac{\partial P}{\partial \theta} \right)_n \delta \theta(k, \omega) \right] - i \left( \frac{4}{3} \eta + \zeta \right) \sum_\beta k^\alpha k^\beta V_l^\beta(k, \omega) + k^\alpha n_0 U(k, \omega),
\end{aligned} \tag{4}$$

$$\omega \left[ \left( \frac{\partial S}{\partial n} \right)_\theta \delta n(k, \omega) + \left( \frac{\partial S}{\partial \theta} \right)_n \delta \theta(k, \omega) \right] = -i \frac{\varkappa}{n_0 \theta_0} k^2 \delta \theta(k, \omega),$$

$$mn_0 \omega V_\tau^\alpha(k, \omega) = -i \eta k^2 V_\tau^\alpha(k, \omega) - n_0 \omega A^\alpha(k, \omega).$$

The solution of (4) gives

$$V_\tau^\alpha(k, \omega) = -\frac{\omega}{m} \frac{A^\alpha(k, \omega)}{\omega + i \frac{\eta}{mn_0} k^2}, \quad (5)$$

$$\delta n(k, \omega) = \frac{n_0 k^2}{m} \frac{\omega + i \frac{c_p}{c_v} D_\theta k^2}{\Omega(k, \omega)} U(k, \omega), \quad (6)$$

$$\delta \theta(k, \omega) = \frac{n_0 k^2}{m} \frac{\omega (\partial S / \partial n)_0 (\partial S / \partial \theta)_n^{-1}}{\Omega(k, \omega)} U(k, \omega), \quad (7)$$

$$V_l^\alpha(k, \omega) = \frac{k^\alpha}{m} \frac{\omega^2 + i \frac{c_p}{c_v} D_\theta k^2 \omega}{\Omega(k, \omega)} U(k, \omega), \quad (8)$$

where

$$\Omega(k, \omega) = \omega^3 + ik^2 \left( D_l + \frac{c_p}{c_v} D_\theta \right) \omega^2 - \left( k^2 C^2 + \frac{c_p}{c_v} k^4 D_\theta D_l \right) \omega - ik^4 C^2 D_\theta, \quad (9)$$

$$D_l = \frac{\frac{4}{3}\eta + \zeta}{mn_0}, \quad D_\theta = \frac{\varkappa}{n_0 c_p}, \quad C^2 = \frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_S.$$

Let us now put

$$U(t, \mathbf{r}) = e^{-i\omega t + \varepsilon t + i\mathbf{k}\mathbf{r}} U(k, \omega) + e^{i\omega t + \varepsilon t - i\mathbf{k}\mathbf{r}} U(-k, -\omega), \quad \mathbf{A}(t, \mathbf{r}) = 0,$$

$$\varepsilon > 0, \quad \varepsilon \rightarrow 0.$$

Then, by the theorem on the variation of the mean, for the quantity  $B(r)$  we have

$$\delta \langle B(k, \omega) \rangle = 2\pi U(k, \omega) \int e^{i\mathbf{k}(\mathbf{r}' - \mathbf{r})} \langle \langle B(r); \Psi^+(\mathbf{r}') \Psi(\mathbf{r}') \rangle \rangle_\omega^r d\mathbf{r}' \quad (10)$$

$$= \frac{2\pi U(k, \omega)}{V} \int e^{i\mathbf{k}(\mathbf{r}'-\mathbf{r})} \langle\langle B(\mathbf{r}); \Psi^+(\mathbf{r}')\Psi(\mathbf{r}') \rangle\rangle_{\omega}^r d\mathbf{r} d\mathbf{r}' = \frac{2\pi U(k, \omega)}{V} \langle\langle B_{\mathbf{k}}; \rho_{-\mathbf{k}} \rangle\rangle_{\omega}^r.$$

If, as  $B_{\mathbf{k}}$ , we take the Fourier images of the operators of particle density, flux density, and internal-energy density,

$$\rho_{\mathbf{k}} = \sum_p a_p^+ a_{p+\mathbf{k}}, \quad j_{\mathbf{k}}^{\alpha} = \sum_p \left( p^{\alpha} + \frac{k^{\alpha}}{2} \right) a_p^+ a_{p+\mathbf{k}},$$

$$\varepsilon_{\mathbf{k}} = \frac{1}{2m} \sum_p p(p+\mathbf{k}) a_p^+ a_{p+\mathbf{k}} + \frac{1}{2V} \sum_{p'+q'=p+\mathbf{k}} v(q'-q) a_p^+ a_q^+ a_{q'} a_{p'},$$

and use (6)–(8), then for the Fourier transforms of the retarded Green functions we obtain the expressions

$$\langle\langle \rho_{\mathbf{k}}; \rho_{-\mathbf{k}} \rangle\rangle_{\omega}^r = \frac{n_0 k^2 V}{2\pi m} \frac{\omega + i \frac{c_p}{c_v} D_{\theta} k^2}{\Omega(k, \omega)},$$

$$\langle\langle j_l^{\alpha}(k); \rho_{-\mathbf{k}} \rangle\rangle_{\omega}^r = \frac{k^{\alpha} n_0 V}{2\pi} \frac{\omega^2 + i k^2 \frac{c_p}{c_v} D_{\theta} \omega}{\Omega(k, \omega)}, \quad (11)$$

$$\langle\langle \varepsilon_{\mathbf{k}}; \rho_{-\mathbf{k}} \rangle\rangle_{\omega}^r = \frac{n_0 k^2 V}{2\pi m} \frac{\varepsilon_0 \omega + i \frac{c_p}{c_v} D_{\theta} k^2 \left[ \varepsilon_0 + \left( \frac{\partial \varepsilon}{\partial n} \right)_{\theta} \right]}{\Omega(k, \omega)},$$

where  $\varepsilon_0$  is the mean internal energy per particle in the state of thermodynamic equilibrium.

To find the tangential component of the tensor Green function current–current, we set

$$A^{\alpha}(t, r) = e^{-i\omega t + \varepsilon t + ikr} A^{\alpha}(k, \omega) + e^{i\omega t + \varepsilon t - ikr} A^{\alpha}(-k, -\omega),$$

$$U(t, r) = 0,$$

where  $A^{\alpha}(k, \omega)$ , as before, are quantities of first order of smallness. Then

$$mn_0 V_{\tau}^{\alpha}(t, r) = \delta \left\langle \frac{i}{2} \left( \frac{\partial \Psi^+}{\partial r^{\alpha}} \Psi - \Psi^+ \frac{\partial \Psi}{\partial r^{\alpha}} \right) \right\rangle - n_0 A^{\alpha}(t, r)$$

and, by virtue of the same theorem on the variation of the mean,

$$mn_0 V_\tau^\alpha(k, \omega) + n_0 A^\alpha(k, \omega) = \frac{2\pi}{mV} \sum_\beta \langle\langle j_\tau^\alpha(k); j_\tau^\beta(-k) \rangle\rangle_\omega^r A^\beta(k, \omega),$$

whence, in accordance with (5),

$$\langle\langle j_\tau^\alpha(k); j_\tau^\beta(-k) \rangle\rangle_\omega^r = -\frac{V}{2\pi} \frac{i\eta k^2}{\omega + i\frac{\eta}{mn_0} k^2} \left( \delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right). \quad (12)$$

To find the correlation functions, we define the corresponding advanced Green functions, which are most simply obtained from (11) by the following reasoning. The formal replacement in (3)  $t \rightarrow -t$ ,  $\mathbf{A} \rightarrow -\mathbf{A}$  is equivalent to changing the signs in front of the coefficients of viscosity and thermal conductivity and, consequently, not to damping but to “pumping” the system, which at  $t = +\infty$  was in a state of thermodynamic equilibrium. For this case, in the theorem on the variation of the mean one must take not the retarded, but the advanced Green function. Thus, to find the advanced Green functions, in (11) and (12) one must make the replacement

$$D_l \rightarrow -D_l, \quad D_\theta \rightarrow -D_\theta, \quad \eta \rightarrow -\eta.$$

Then

$$\begin{aligned} \langle\langle \rho_k; \rho_{-k} \rangle\rangle_\omega^a &= \frac{n_0 k^2 V}{2\pi m} \frac{\omega - i\frac{c_p}{c_v} D_\theta k^2}{\Omega^*(k, \omega)}, \\ \langle\langle j_l^\alpha(k); \rho_{-k} \rangle\rangle_\omega^a &= \frac{k^\alpha n_0 V}{2\pi} \frac{\omega^2 - ik^2 \frac{c_p}{c_v} D_\theta \omega}{\Omega^*(k, \omega)}, \\ \langle\langle \varepsilon_k; \rho_{-k} \rangle\rangle_\omega^a &= \frac{n_0 k^2 V}{2\pi m} \frac{\varepsilon_0 \omega - i\frac{c_p}{c_v} D_\theta k^2 \left[ \varepsilon_0 + \left( \frac{\partial \varepsilon}{\partial n} \right)_0 \right]}{\Omega^*(k, \omega)}, \\ \langle\langle j_\tau^\alpha(k); j_\tau^\beta(-k) \rangle\rangle_\omega^a &= \frac{V}{2\pi} \frac{i\eta k^2}{\omega - i\frac{\eta}{mn_0} k^2} \left( \delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right). \end{aligned} \quad (13)$$

Now, according to the well-developed procedure <sup>(2)</sup>, it is not difficult to find the corresponding Fourier transforms of the correlation functions:

$$J_{\rho\rho}(k, \omega) = \frac{n_0 k^4 \omega \left[ D_l \omega^2 + k^2 C^2 D_\theta \left( \frac{c_p}{c_v} - 1 \right) + \left( \frac{c_p}{c_v} \right)^2 k^4 D_\theta D_l \right] V}{\pi m |\Omega(k, \omega)|^2 (e^{\omega/\theta} - 1)},$$

$$J_{i,\rho}^\alpha(k, \omega) = \frac{k^\alpha k^2 n_0 \omega^2 \left[ D_l \omega^2 + k^2 C^2 D_\theta \left( \frac{c_p}{c_v} - 1 \right) + \left( \frac{c_p}{c_v} \right)^2 k^4 D_\theta^2 D_l \right] V}{\pi m |\Omega(k, \omega)|^2 (e^{\omega/\theta} - 1)}, \quad (14)$$

$$J_{\tau_i \tau}^{\alpha, \beta}(k, \omega) = \frac{\eta k^2 \omega V}{\pi \left[ \omega^2 + \left( \frac{\eta}{m n_0} k^2 \right)^2 \right] (e^{\omega/\theta} - 1)} \left( \delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right).$$

$$I_{\varepsilon\rho}(k, \omega) = \frac{n_0 k^4 \omega V}{\pi m |\Omega(k, \omega)|^2 (e^{\omega/\theta} - 1)} \left\{ \left[ \varepsilon_0 D_l - \frac{c_p}{c_v} D_\theta \left( \frac{\partial \varepsilon}{\partial n} \right)_\theta \right] \omega^2 \right. \\ \left. + \left( \frac{c_p}{c_v} - 1 \right) k^2 C^2 D_\theta + \frac{c_p}{c_v} D_\theta k^2 C^2 \left( \frac{\partial \varepsilon}{\partial n} \right)_\theta \right. \\ \left. + \left( \frac{c_p}{c_v} \right)^2 D_\theta^2 D_l^4 \left( \varepsilon_0 + \left( \frac{\partial \varepsilon}{\partial n} \right)_\theta \right) \right\}.$$

In the approximation of weak dissipation, expressions (14) coincide with the results of <sup>(1)</sup>, and (11) with the results of <sup>(5)</sup>.

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