On the First Transcendental Integrals of the Equations of Motion of a Heavy Rigid Body About a Fixed Point

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Abstract

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On the First Transcendental Integrals of the Equations of Motion of a Heavy Rigid Body About a Fixed Point

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As is known, the equations of motion of a heavy rigid body about a fixed point

$$A\,dp/dt + (C-B)qr = Mg(y_0\gamma'' - z_0\gamma'), \qquad d\gamma/dt = r\gamma' - q\gamma'' \tag{1}$$

$$(ABC,\ pqr,\ x_0y_0z_0,\ \gamma\gamma'\gamma''),$$

besides three algebraic first integrals (energy, areas, and cosines), possess two transcendental first integrals independent of time,

$$H_i(p,q,r,\gamma,\gamma',\gamma'') = c_i \qquad (i = 4, 5), \tag{2}$$

of which the fourth reduces to an algebraic integral only in the three cases: Euler, Lagrange, and Kovalevskaya ([1,2]).

In view of the great difficulties, up to the present time the fourth transcendental integral has not been found in a single case, and its properties, with rare exceptions, have not been studied.

In the present paper, on the basis of the results of papers ([3–6]), one new property of the integrals (2) is stated in the form of a theorem; this property may serve for determining the structure of time-independent first transcendental integrals of system (1), and a second theorem is formulated concerning integrals of system (1) linear in p and q.

1. In paper ([3]), by a change of variables, equations (1) in the general case were reduced to the dimensionless equations

$$\dot{p}_1 + (C-B)A^{-1}q_1r_1 = \mu CA^{-1}(y_0'\gamma_1'' - z_0'\gamma_1'), \qquad \dot{\gamma}_1 = r_1\gamma_1' - \mu q_1\gamma_1'',$$
$$\dot{q}_1 + (A-C)B^{-1}p_1r_1 = \mu CB^{-1}(z_0'\gamma_1 - x_0'\gamma_1''), \qquad \dot{\gamma}_1' = \mu p_1\gamma_1'' - r_1\gamma_1,$$
$$\dot{r}_1 = \mu^2[-(B-A)C^{-1}p_1q_1 + x_0'\gamma_1' - y_0'\gamma_1], \quad \dot{\gamma}_1'' = \mu(q_1\gamma_1 - p_1\gamma_1'),$$

$$(du/dt) = \dot{u} \tag{3}$$

with three algebraic first integrals (apart, of course, from two transcendental ones) of the form

$$r_1^2 = 1 - \mu^2\left[\frac{A}{C}p_1^2 + \frac{B}{C}q_1^2 - 2(x_0'\gamma_1 + y_0'\gamma_1' + z_0'\gamma_1'')\right] + \mu^2 c_1^*,$$

$$r_1\gamma_1' = 1 - \mu\left(\frac{A}{C}p_1\gamma_1 + \frac{B}{C}q_1\gamma_1'\right) + \mu c_2^*, \tag{4}$$

$$H_3 = \gamma_1^2 + \gamma_1'^2 + \gamma_1''^2 = c_3^*, \qquad 1 < c_3^* < \infty$$

(c_1^*, c_2^*, c_3^* are constants, μ is a small parameter).

With the help of the first two integrals (4), system (3) was reduced to a quasi-linear autonomous system with two degrees of freedom

$$d^2p_1/d\tau^2 + \omega^2 p_1 = \mu F_1^*(p_1, dp_1/d\tau, \gamma_1, d\gamma_1/d\tau, \mu), \qquad \omega^2 = \frac{(A-C)(B-C)}{AB},$$

$$d^2\gamma_1/d\tau^2 + \gamma_1 = \mu F_2^*(p_1, dp_1/d\tau, \gamma_1, d\gamma_1/d\tau, \mu), \tag{5}$$

possessing one algebraic and two transcendental first integrals

$$H_3^* = \gamma_1^2 + (d\gamma_1/dt)^2 + 1 + \mu(\ldots) = c_3^*,$$
$$H_i^*(p_1, dp_1/dt, \gamma_1, d\gamma_1/dt, \mu) = c_i^* \quad (i = 4, 5). \tag{6}$$

The periodic solutions of system (5) that have been found and depend only on one arbitrary principal amplitude $\rho = \sqrt{c_3^* - 1}$ ($\rho = M_3 = (1 - \gamma_0''^2)^{1/2}\gamma_0''^{-1}$ in the notation of paper [3]) made it possible to indicate [3–5] periodic solutions, of period T, $p_1(\tau, \mu), \ldots, \gamma_1''(\tau, \mu)$ of system (3), which depend on three arbitrary constants τ_0, μ, ρ.

Next, rewriting the first three integrals (6) in the form of differences

$$H_i^*[p_1(T,\mu),\dot p_1(T,\mu),\gamma_1(T,\mu),\dot\gamma_1(T,\mu),\mu]-$$
$$-H_i^*[p_1(0,\mu),\dot p_1(0,\mu),\gamma_1(0,\mu),0,\mu]=0 \quad (i=3,4,5),$$

we obtain, with the aid of the notation

$$p_1(T,\mu)-p_1(0,\mu)=\psi_1, \qquad \dot p_1(T,\mu)-\dot p_1(0,\mu)=\psi_2,$$

$$\gamma_1(T,\mu)-\gamma_1(0,\mu)=\psi_3, \qquad \dot\gamma_1(T,\mu)=\psi_4,$$

the relations

$$H_i^*[p_1(0,\mu)+\psi_1,\dots,\psi_4,\mu]-H_i^*[p_1(0,\mu),\dots,0,\mu]$$
$$=H_i^{(1)}(\psi_1,\psi_2,\psi_3,\psi_4)=0.$$

On the basis of (6) one can show that the existence of three first integrals (6) of the quasilinear system (5) entails the conversion into arbitrary constants of at least as many principal amplitudes in the periodic solutions of this system as the rank of the Jacobi matrix

$$M=\left\|\partial H_i^{(1)}/\partial\psi_j\right\|_{\psi_1=\dots\psi_4=\mu=0} \quad (i=3,4,5;\ j=1,2,3).$$

Since the periodic solutions found for system (5) depend only on one arbitrary constant ρ, the rank of the matrix M cannot be greater than unity in a neighborhood of the corresponding initial conditions. Hence, on the basis of the relations, which hold under the conditions $\psi_1=\dots=\psi_4=\mu=0$,

$$\partial H_3^{(1)}/\partial\psi_1=\partial H_3^{(1)}/\partial\psi_2=0, \qquad \partial H_3^{(1)}/\partial\psi_3=2\gamma_1(0,0)=2\rho\neq 0,$$

it follows that

$$\left[(\partial H_i^{(1)}/\partial\psi_1)^2+(\partial H_i^{(1)}/\partial\psi_2)^2\right]_{\psi_1=\dots=\psi_4=\mu=0}=0 \quad (i=4,5).$$

Moreover, the integrals (2), rewritten on the basis of (4) in the form $H_i^{(2)}(p_1,q_1,\gamma_1)=c_i^{(2)}$ $(i=4,5)$, cannot depend only on one of their arguments (^{2}), since the contrary assumption leads to the existence of the corresponding first integral $p_1=\text{const.}$, $q_1=\text{const.}$, $\gamma_1=\text{const.}$, which, under the conditions considered, $A\geq B>C$ or $A\leq B<C$, does not occur.

2. Choosing, for convenience, the system of units in such a way that $Mg/C = 1$, we introduce into equations (1) an arbitrary parameter μ by the formulas

$$p = \mu p_1, \qquad q = \mu q_1, \qquad r = r_1, \qquad \gamma = \gamma_1, \qquad \gamma' = \gamma'_1, \qquad \gamma'' = \gamma''_1,$$

$$x_0 = \mu^2 x'_0, \qquad y_0 = \mu^2 y'_0, \qquad z_0 = \mu^2 z'_0. \tag{7}$$

Then system (1) will formally pass into system (3), and any of the integrals (2) will pass into an integral of the form

$$H(p_1, q_1, r_1, \gamma_1, \gamma'_1, \gamma''_1, \mu) = c. \tag{8}$$

This integral will be a holomorphic function of all its arguments in a neighborhood of the point $(p_{10}, q_{10}, 1, \gamma_{10}, 0, 1, 0)$, where the initial values $p_{10}, q_{10}, \gamma_{10}$ are arbitrary and finite and $\gamma_{10} \neq 0$. Substituting r_1 and γ''_1 into the integral (8)

from (4) and expanding it in a series in powers of μ, we obtain the relation

$$H_0(p_1, q_1, 1, \gamma_1, 0, 1, 0) + \mu(...) = c_0,$$

in which, by multiplying by μ to the appropriate power and transforming the arbitrary constant c_0, we have $H_0 \neq \text{const}$.

On the basis of the results set forth above, the following can be proved.

Theorem 1. *For any time-independent transcendental first integral (2) of system (1), the expression H_0 is such that the quantity*

$$J = (\partial H_0/\partial p_1)^2 + (\partial H_0/\partial q_1)^2 \qquad (J \neq 0)$$

vanishes for the values of p_1, q_1 equal to $p_1(0,0), q_1(0,0)$, obtained from the corresponding formulas of papers (3–5), and for the value of γ_1 equal to the arbitrary constant ρ.

As an example one may consider the Kovalevskaya case ($A = B = 2C$, $y_0 = z_0 = 0$). In this case, for instance, the conditions obtained in paper (5) are satisfied, and the fourth transcendental integral (2) becomes the well-known Kovalevskaya integral, for which

$$H_0 = (p_1^2 - q_1^2 + x'_0\gamma_1)^2 + 4p_1^2 q_1^2.$$

From the preceding theorem there follows, as a consequence,

Theorem 2. *In the general case there cannot exist a time-independent transcendental first integral of system (1) that is linear with respect to p and q.*

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Note: Figure translations are in progress. See original paper for figures.