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MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE EMBEDDING OF DISCRETE FIELDS IN CONNECTED ONES

(Presented by Academician A. I. Mal' tsev on 5 X 1965)

We begin with an auxiliary construction. Denote by C the interval of the real axis $[0, 1]$; by Ω, the set of all equivalence relations on C such that every class is finite and all classes, except for a finite number, are singleton. We shall denote elements of Ω by lowercase letters of the Greek alphabet. We shall say that $\alpha \geqslant \beta$ if $a\beta b$ implies $a\alpha b$. As is easy to verify, Ω, with respect to this ordering, is a lattice.

By the **length** of an α-class M we shall mean $\max M - \min M$, and denote it by $\operatorname{long} M$. By the **length** of α we shall mean the sum of the lengths of the α-classes (all of them, except for a finite number, are equal to 0).

Lemma 1. Let M be an $\alpha \cup \beta$-class; $P_1, P_2, \dots, P_m$ and $Q_1, Q_2, \dots, Q_n$ all the nonsingleton α- and β-classes contained in it, respectively. Then

$$\operatorname{long} M \leqslant \sum_{i=1}^{m} \operatorname{long} P_i + \sum_{i=1}^{n} \operatorname{long} Q_i .$$

Proof. Suppose the contrary. Denote $[\min M; \max M] = \mathrm{M}$, $[\min P_i; \max P_i] = \Pi_i$, and $[\min Q_i; \max Q_i] = \Psi_i$, where the lengths of M, Π_i, and Ψ_i are equal to the lengths of the corresponding classes. From our supposition it follows that there exists $x \in \mathrm{M}$ such that

$$x \notin \bigcup_{i=1}^{m} \Pi_i \cup \bigcup_{i=1}^{n} \Psi_i .$$

Define $a\gamma b$ in each of the following three cases:

a) $a(\alpha \cup \beta)b,\ a \notin M,\ b \notin M;$

b) $a, b \in M,\ a < x,\ b < x;$

c) $a, b \in M,\ a > x,\ b > x.$

It is not hard to verify that $\gamma \in \Omega$, $\gamma \geqslant \alpha$, $\gamma \geqslant \beta$, $\gamma \leqslant \alpha \cup \beta$, whence $\gamma = \alpha \cup \beta$. But $\min M \, \gamma \, \max M$ does not hold, whereas $\min M (\alpha \cup \beta) \max M$ does hold, and the lemma is proved.

Since every α-class and every β-class is contained in only one $\alpha \cup \beta$-class, we have

$$\operatorname{long}(\alpha \cup \beta) \leqslant \operatorname{long} \alpha + \operatorname{long} \beta. \tag{1}$$

Denote by α^* the following transformation of C into itself:

$$x\alpha^* = \min_{x\alpha y} y.$$

Lemma 2. If $\alpha \leqslant \beta$, then $\alpha^* \beta^* = \beta^*$.

Proof. From $x\alpha^*\alpha x$ we obtain

$$x\alpha^*\beta^* = \min_{x\alpha^*\beta y} y = \min_{x\beta y} y = x\beta^*.$$

Let P be an arbitrary field. Denote by P' the ring of polynomials over P in the indeterminates s_x, where x ranges over C. Denote by P^0 the field of rational functions over P in the same indeterminates. Denote by α' the transformation of the ring P' consisting in replacing, in the notation of a polynomial, the indeterminates s_x by $s_{x\alpha^*}$. It is easy to verify that α' is a homomorphism.

By α^0 we shall denote the transformation, defined on those $f \in P^0$ for which the denominator of f in the canonical representation is not mapped by the transformation α' to 0, and consisting in the following: if f is written in the form $\sum_{i=1}^{n} \frac{a_i}{b_i}$, where $a_i, b_i \in P'$ and $b_i\alpha' \neq 0$, then set

$$f\alpha^0 = \sum_{i=1}^{n} \frac{a_i\alpha'}{b_i\alpha'}.$$

Let us prove the correctness of the last definition. Suppose

$$\sum_{i=1}^{m} \frac{a_i}{b_i} = \sum_{i=1}^{n} \frac{c_i}{d_i}.$$

Then

$$\sum_{i=1}^{m} \left(a_i \prod_{\substack{j=1 \\ j \neq i}}^{m} b_j \prod_{k=1}^{n} d_k \right) = \sum_{i=1}^{n} \left(c_i \prod_{\substack{k=1 \\ k \neq i}}^{n} d_k \prod_{j=1}^{m} b_j \right),$$

whence we have the same equality with α' appended after each polynomial, from which, after division by the product of all $b_j\alpha'$ and $d_k\alpha'$, we obtain the equality

$$\sum_{i=1}^{m} \frac{a_i\alpha'}{b_i\alpha'} = \sum_{i=1}^{n} \frac{c_i\alpha'}{d_i\alpha'}.$$

Let $p \in P$. Then

$$p\alpha^0 = \left(\frac{p}{1}\right)\alpha^0 = \frac{p\alpha'}{1\alpha'} = \frac{p}{1} = p.$$

The **length of a polynomial** f ($\operatorname{long} f$) will mean

$$\min_{f\alpha'=0} [\inf \operatorname{long} \alpha;\ 1]^*.$$

The length of a polynomial not equal to 0 is, obviously, not less than the minimal distance between the lower indices of the elements s_x occurring in the expression of this polynomial, i.e., from $\operatorname{long} f = 0$ it follows that $f = 0$. We note one more obvious property of length:

$$\operatorname{long}(fg) = \min[\operatorname{long} f; \operatorname{long} g]. \tag{2}$$

The **weight of the expression** $\sum_{i=1}^{n} \frac{a_i}{b_i}$, where $a_i, b_i \in P'$, $b_i \neq 0$, will mean

$$\sum_{i=1}^{n} \frac{\operatorname{long} a_i}{\operatorname{long} b_i}.$$

The **norm of a rational function** f will mean the exact lower bound of the weights of its expressions, and we shall denote it by $\|f\|$. Let us study the properties of this norm. The first two properties need no proof:

1°. $\|-f\| = \|f\|$.

2°. $\|f+g\| \leq \|f\| + \|g\|$.

Let

$$f = \sum_{i\in I} \frac{a_i}{b_i} \quad \text{and} \quad g = \sum_{j\in J} \frac{a_j}{b_j}.$$

Denote the weight of the first expression by p, and that of the second by q. Order $I \cup J$ arbitrarily, but so that from $k < l$ (where $k, l \in I \cup J$) it would follow that $\operatorname{long} b_k \leq \operatorname{long} b_l$. We have

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$$fg = \sum_{k\in I}\left(\frac{a_k}{b_k}\sum_{\substack{l\in J\\ l>k}}\frac{a_l}{b_l}\right) + \sum_{k\in J}\left(\frac{a_k}{b_k}\sum_{\substack{l\in I\\ l>k}}\frac{a_l}{b_l}\right).$$

After bringing the expression in parentheses to a common denominator, the length of its denominator will, by (2), be equal to $\operatorname{long} b_k$, while the length of the numerator, by the same equality, will be no greater than $\operatorname{long} a_k$; i.e., the weight of such an expression is less than or equal to

$$\sum_{k\in I\cup J}\frac{\operatorname{long} a_k}{\operatorname{long} b_k} = p+q,$$

whence

$$3^\circ.\ \|fg\| \le \|f\| + \|g\|.$$

Let f and g be arbitrary elements of P^0, let $\frac{a}{b}$ be an irreducible expression of the first of them and $\sum_{i\in I}\frac{a_i}{b_i}$, where $a_i, b_i \in P'$, an arbitrary expression of the second—

∗ Here the infimum of the empty set is understood to be $+\infty$.

Therefore. By (2) we have

$$\frac{\operatorname{long}(aa_i)}{\operatorname{long}(bb_i)} \le \frac{\operatorname{long} a_i}{\operatorname{long} b\ \operatorname{long} b_i},$$

whence

$$\sum_{i\in I}\frac{\operatorname{long}(aa_i)}{\operatorname{long}(bb_i)} \le \frac{1}{\operatorname{long} b}\sum_{i\in I}\frac{\operatorname{long} a_i}{\operatorname{long} b_i}.$$

In view of the arbitrariness of the expression for g, we have

$$\|fg\| \le \frac{1}{\operatorname{long} b}\|g\|,$$

i.e.

4°. $\|fg\| \le C_f\|g\|$, where C_f depends only on f.

Let $\|f\| < 1$. Then there exists a representation

$$f = \sum_{i=1}^{n}\frac{a_i}{b_i},$$

where $a_i, b_i \in P'$, such that

$$\sum_{i=1}^{n} \frac{\operatorname{long} a_i}{\operatorname{long} b_i} = p < 1.$$

Hence $\operatorname{long} a_i < 1$ for all i. Thus there exist α_i such that $a_i \alpha_i' = 0$ and

$$\operatorname{long} \alpha_i < \frac{\operatorname{long} a_i}{p}.$$

We may assume that, for $i < j$, one has $\operatorname{long} b_i \leq \operatorname{long} b_j$; otherwise we interchange the summands. Put

$$\bigcup_{i=1}^{k} \alpha_i = \beta_k; \qquad \beta_1^0 \beta_2^0 \ldots \beta_k^0 = \Gamma_k.$$

Since from $\xi^* \eta^* = \zeta^*$ it follows that $\xi' \eta' = \zeta'$, by Lemma 2 we obtain

$$\beta_1' \beta_2' \ldots \beta_k' = \beta_k' * .$$

By induction it is easily proved that

$$\Gamma_k = \sum_{i=k+1}^{n} \frac{a_i \beta_k'}{b_i \beta_k'}$$

(the transformation β_{k+1}' is applicable to $f\Gamma_k$, because, by (1), we have

$$\operatorname{long} \beta_{k+1} \leq \sum_{i=1}^{k} \operatorname{long} \alpha_i < \frac{1}{p} \sum_{i=1}^{k} \operatorname{long} a_i \leq \frac{1}{p} \operatorname{long} b_k \sum_{i=1}^{k} \frac{\operatorname{long} a_i}{\operatorname{long} b_i} \leq \operatorname{long} b_k \leq \operatorname{long} b_l,$$

where $l > k$). Hence $f\Gamma_n^* = 0$. If $r \in P$, then $r\Gamma_n = r$, i.e. from $\|f\| < 1$ it follows that $f \in P \setminus \{0\}$.

Denote by K the set of those $f \in P^0$ for which $\|f\| = 0$. By $1^\circ, 2^\circ$ and 4°, we have that K is an ideal. But P^0 is a field, $\|0\| = 0$ and $\|1\| \geq 1$. Hence $K = \{0\}$. We have obtained the last property of the norm:

5°. $\|f\| = 0$ if and only if $f = 0$.

Definition. A topological ring is called a **semitopological field** if it is a field.

Denote by U_a, where a is a positive real number, the set of all f such that $\|f\| < a$. Then from 1°–5° it follows that the U_a form a fundamental system of neighborhoods of 0 of a certain semitopologization of the field P^0, i.e. P^0 is a semitopological field. Denote by ξ_{ab}, where $a, b \in C$, the equivalence relation with the unique non-one-element class $\{a, b\}$. Then

$$\operatorname{long} \xi_{ab} = |a - b|$$

and

$$(s_a - s_b)\xi'_{ab} = s_{\min[a,b]} - s_{\min[a,b]} = 0,$$

i.e.

$$\|s_a - s_b\| \leq |a - b|;$$

in other words, the topology on s_C is weaker than or the same as the one induced from C under the mapping $x \mapsto s_x$. But the topology of C is connected. Hence s_C is all the more connected; and if in a semitopological field there exists a connected non-one-element set, then the field itself is connected, since by a nondegenerate linear transformation any pair of distinct points is carried into any pair of distinct points, while a nondegenerate linear transformation in a semitopological field is a homeomorphism.

We shall give a new proof of Theorem 9 from (1):

Lemma 3. *The topology of a semitopological field is weakened to the topology of a topological field.*

Proof. Choose any fundamental system of symmetric (i.e. containing $-x$ together with x) neighborhoods of 0 of the semitopological of the topological field $\{U_\alpha\}$. Consider the system of symmetric sets

$$\frac{U_\alpha}{1 + (U_\alpha \setminus \{-1\})}.$$

Having verified for it the required properties, we are convinced that this is a fundamental system of neighborhoods of 0 for some field topologization. As an example, let us verify separability. Let $x \neq 0$; then there exists $U_\alpha \not\ni x$. There exists U_β such that $U_\beta + U_\beta \subseteq U_\alpha$. There exists U_γ such that $xU_\gamma \subseteq U_\beta$ and $U_\gamma \subseteq U_\beta$. Then $U_\gamma + xU_\gamma \not\ni x$, whence, by symmetry,

$$(x + xU_\gamma) \cap U_\gamma = \emptyset,$$

or

$$\{x\} \cap \frac{U_\gamma}{1 + (U_\gamma \setminus \{-1\})} = \emptyset,$$

as was required to prove. Since

$$\frac{U_\alpha}{1 + (U_\alpha \setminus \{-1\})} \supset U_\alpha,$$

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* But $\Gamma_k \neq \beta_k^0$, in general, since from $\xi^*\eta^* = \zeta^*$ it does not follow that $\xi^0\eta^0 = \zeta^0$ (the domain of definition may change).

the new topology is the same or weaker.

Apply the lemma just proved to P^0, to the system $\{U_\alpha\}$. We obtain some field topologization τ. It is connected, since it is weaker than, or the same as, the original one. In conclusion, let us show that P is discrete in P^0_τ. From (2) it follows that, for $a \in P \setminus \{0\}$ and $f \in P'$, we have

$$\operatorname{long}(af) = \min[\operatorname{long} a; \operatorname{long} f] \leq \operatorname{long} f = \operatorname{long}(a^{-1}af) = \min[\operatorname{long} a^{-1}; \operatorname{long}(af)] \leq \operatorname{long}(af),$$

i.e. $\operatorname{long} f = \operatorname{long}(af)$, whence it follows that $\|ax\| = \|x\|$, if $a \in P \setminus \{0\}$. Hence

$$(P \setminus \{0\})U_{1/2} \subseteq U_{1/2}.$$

By the property obtained in the proof of 5°,

$$(P \setminus \{0\}) \cap U_1 = \emptyset,$$

and hence

$$\frac{U'_{1/2}}{1 + U_{1/2}} \cap (P \setminus \{0\}) = \emptyset,$$

and, therefore, P is discrete. Thus, the following has been proved.

Theorem. *Every discrete field can be embedded in a connected one.*

Corollary. *There exists a connected field of any characteristic.*

The latter solves the problem posed by I. R. Shafarevich. I take this opportunity to express my gratitude to Prof. L. A. Skornyakov for valuable comments on the manuscript.

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Note: Figure translations are in progress. See original paper for figures.

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