GENERAL ADDITIONAL CONDITIONS FOR A BOUNDARY-VALUE PROBLEM WITH OBLIQUE DERIVATIVE

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GENERAL ADDITIONAL CONDITIONS FOR A BOUNDARY-VALUE PROBLEM WITH OBLIQUE DERIVATIVE

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1. Let E be a plane domain bounded by a closed contour L, and let $v(z)$ be a smoothly varying vector field on L. We shall call a point $\gamma \in L$ exceptional if the projection of the vector $v(z)$ onto the inward normal to L changes sign at the point γ. We shall say that a function $u(z)$ satisfies the boundary condition $\mathcal{A}$ if at every nonexceptional point z of the contour L the derivative of u in the direction $v(z)$ is equal to zero. We are interested in solutions of the heat equation $du_t(z)/dt = \Delta u_t(z)$ in the domain E, satisfying the initial condition $u_0(z) = f(z)$ and the boundary condition $\mathcal{A}$. More precisely, our problem is to describe the general form of the additional conditions which, together with the initial and boundary conditions, determine a unique solution $u_t(z)$ of the heat equation, with: a) $u_t(z) \geq 0$ if $f(z) \geq 0$; b) $\|u_t\| \leq \|f\|$ (by $\|f\|$, here and below, is meant $\sup|f(z)|$ on the union E^* of the domain E and the set of all nonexceptional points of the contour L).

In the language of the theory of semigroups of operators, the problem can be formulated as follows. Let $B(E^*)$ be the set of all bounded Borel functions on E^*, and let D be the totality of all functions in $B(E^*)$ having continuous, in the Hölder sense, first partial derivatives in E^*, continuous, in the Hölder sense, second partial derivatives in E, and satisfying the boundary condition $\mathcal{A}$. Consider, on the domain D, the Laplace operator Δ, and denote by $\mathfrak{A}$ its w-closure in $B(E^*)$ *. We shall call an $\mathfrak{A}$-semigroup a one-parameter semigroup T_t of linear operators in the space $B(E^*)$ satisfying the conditions: a) $T_t f \geq 0$ for $f \geq 0$; b) $\|T_t f\| \leq \|f\|$; c) the infinitesimal operator A of the semigroup T_t is a restriction of the operator $\mathfrak{A}$; d) the w-closure of the set $B_0(E^*)$ coincides with $B(E^*)$ **. It is required to describe all $\mathfrak{A}$-semigroups.

Finally, let us indicate the probabilistic interpretation of our problem. The heat equation together with the boundary condition $\mathcal{A}$ defines in the domain E a process of Brownian motion with reflection from the boundary in the direction of the vector field $v(z)$ (or $-v(z)$, if $v(z)$ is directed outside the domain E). The behavior of the trajectory after it hits an exceptional boundary point is not determined thereby. The problem is to describe all possible types of such behavior.

2. Suppose that we move along the contour L, passing through an exceptional point γ in the direction of the vector $v(\gamma)$, and at the same time observe the projection of the vector $v(z)$ onto the inward normal to the contour L at the point z. Put $\gamma \in \Gamma_+$ if this projection changes sign from plus to minus, and $\gamma \in \Gamma_-$ if the sign changes from minus to plus. Put $\Gamma = \Gamma_+ \cup \Gamma_-$ (this is the set of all exceptional points).

* We say that $f_n \xrightarrow{w} f$ if $f_n(z) \to f(z)$ for all $z \in E^*$ and the sequence $\|f_n\|$ is bounded. The operator $\mathfrak{A}$ is called w-closed if from $f_n \xrightarrow{w} f$, $\mathfrak{A} f_n \xrightarrow{w} F$ it follows that $F = \mathfrak{A} f$. By the w-closure of an operator we mean its minimal w-closed extension.

** By $B_0(E^*)$ is denoted the set of all elements $f \in B(E^*)$ such that $\|T_t f - f\| \to 0$ as $t \downarrow 0$.

It is useful to "split" each point $\gamma \in \Gamma$ into two points $\gamma+$, $\gamma-$. We denote the union of all such pairs by Π. The decomposition of Γ into Γ_+ and Γ_- corresponds to a decomposition of Π into Π_+ and Π_-. If F is a function on E^*, then by $F(\gamma+)$, $F(\gamma-)$ we mean its limits when z tends to γ along the contour L, respectively from the positive and the negative side. It is proved that if $F \in D_{\mathfrak{A}}$, then the limiting values $F(\gamma+)$, $F(\gamma-)$ exist for all $\gamma \in \Gamma$.

To each $a \in \Pi_+$ there corresponds, and moreover only one, bounded harmonic function $p_a(z)$ satisfying the boundary condition $\mathfrak{A}$ and such that $p_a(a) = 1$ and $p_a(\beta) = 0$ for $\beta \in \Pi_+$, $\beta \neq a$. (If, for example, $a = \gamma+$, then $p_a(z)$ is the probability that the trajectory issuing from z approaches γ, touching L from the positive side at γ.)

3. Suppose the following are given: 1) a partition of the set Π_+ into classes; the collection of these classes is denoted by Ω; 2) for each $\omega \in \Omega$, a set of nonnegative constants c_ω, σ_ω, $b_{\omega\gamma}$ ($\gamma \in \Gamma_+$), and a measure ν_ω in the space $\mathcal{E} = E^* \cup \Pi_- \cup \Omega$.

We denote by E^*_ε the collection of all points z of the set E^* for which $\rho(z, \Gamma_+) \geq \varepsilon^*$; by $\chi_{\gamma,\varepsilon}$ the function equal to $|z-\gamma|^2$ for $|z-\gamma| < \varepsilon$ and to zero for $|z-\gamma| \geq \varepsilon$. Let $t = \rho(z, L)$. We put $\omega \in \Omega'$ if $\sigma_\omega = 0$, $b_{\omega\gamma} = 0$ for all $\gamma \in \Gamma_+$, and $\nu_\omega(\mathcal{E}) < \infty$.

We shall assume that, for every $\omega \in \Omega$, the following requirements are satisfied:

a) $\nu_\omega(E^*_\varepsilon) < \infty$, if $\varepsilon > 0$;

b) $(t, \nu_\omega) < \infty$**; for every $\gamma \in \Gamma_+$ and for sufficiently small $\varepsilon > 0$,

$$(\chi_{\gamma,\varepsilon}, \nu_\omega) < \infty; \qquad (p_\alpha, \nu_\omega) < \infty \quad \text{for } \alpha \in \omega;$$

c) $\nu_\omega(\omega) = 0$;

d) $b_{\omega\gamma} = 0$, if at least one of the points $\gamma+$, $\gamma-$ does not belong to ω;

e) at least one of the numbers $\nu_\omega(\mathcal{E})$, $b_{\omega\gamma}$ $(\gamma \in \Gamma_+)$, c_ω, σ_ω is positive.

We shall agree to say that a function F satisfies the supplementary condition $\mathfrak{U}$ if $F \in D_{\mathfrak{A}}$ and, for all $\omega \in \Omega$,

$$(F - F(\omega), \nu_\omega) + \sum_{\gamma \in \Gamma_+} b_{\omega\gamma} \frac{\partial F}{\partial n}(\gamma) - c_\omega F(\omega) - \sigma_\omega \mathfrak{A} F(\omega) = 0, \tag{1}$$

where $\partial F / \partial n$ is the derivative of F in the direction of the inner normal to L, $F(\omega) = F(a)$ and $\mathfrak{A}F(\omega) = \mathfrak{A}F(a)$ (a is an arbitrary point of the class ω). Formula (1) implicitly contains the assumptions that, when $\sigma_\omega > 0$, the values $\mathfrak{A}F(a)$ are defined and are the same for all a belonging to the class ω. We shall say that the semigroup of operators in the space $B(E^*)$ satisfies the supplementary condition $\mathfrak{U}$ if this condition is satisfied by all functions in the domain of definition of its infinitesimal operator.

4. The solution of the problems posed in §1 is given by the following theorem:

Theorem 1. *Every $\mathfrak{A}$-semigroup satisfies some condition $\mathfrak{U}$, and*

$$\sigma_\omega = 0 \quad \text{for all } \omega \in \Omega. \tag{2}$$

An arbitrary supplementary condition $\mathfrak{U}$ for which (2) holds uniquely determines some $\mathfrak{A}$-semigroup.

A natural question arises: what meaning do the conditions $\mathfrak{U}$ have in the case when relation (2) is not satisfied? In this case it is necessary to enlarge the phase space E^* by adjoining to it all points ω for which $\sigma_\omega > 0$. We denote the set of such points by $\widetilde{\Omega}$ and put $\tilde{\mathcal{E}} = E^* \cup \Omega$. For any function F in the space $\tilde{\mathcal{E}}$ we denote

* By $\rho(z, M)$ is denoted the distance from the point z to the set M.

** By (f, ν) is denoted the integral of the function f with respect to the measure ν (over the whole space $\mathcal{E}$).

via F_0 its restriction to the space E^*. Put $F \in \widetilde{B} = B(\tilde{\mathcal{E}})$, if $F_0 \in B(E^*)$. Define in the space $\widetilde{B}$ the operator $\widetilde{\mathfrak{A}}$ by the formulas

$$\widetilde{\mathfrak{A}}F(z) = \mathfrak{A}F(z) \quad \text{for } z \in E^*,$$

$$\widetilde{\mathfrak{A}}F(\omega) = \frac{1}{\sigma_\omega} \left\{ (F - F(\omega), v_\omega) + \sum_{\gamma \in \Gamma_+} b_{\omega\gamma} \frac{\partial F}{\partial n}(\gamma) - c_\omega F(\omega) \right\} \quad \text{for } \omega \in \overline{\Omega}. \tag{3}$$

The domain of definition of $\widetilde{\mathfrak{A}}$ is the set $D_{\widetilde{\mathfrak{A}}}$ of all functions $F \in \widetilde{B}$ for which $F_0 \in D_{\mathfrak{A}}$ and the right-hand side of (3) has meaning. We shall say that F

satisfies the additional condition $\mathcal{U}$ if $F \in D_{\widetilde{\mathfrak{A}}}$ and, for all $\omega \in \Omega$, relation (1) holds with $\mathfrak{A}$ replaced by $\widetilde{\mathfrak{A}}$. In this case the value $F(\omega)$ must coincide with the limiting value $F(\gamma+)$ ($F(\gamma-)$), if $\gamma+ \in \omega$ (respectively, $\gamma- \in \omega$). (We note that for $\omega \in \Omega$ equality (1) holds automatically by virtue of the definition of $\widetilde{\mathfrak{A}}$.)

Theorem 2. *For every additional condition $\mathcal{U}$ one can construct, and moreover only one, semigroup of operators T_t in the space $\tilde{\mathcal{E}}$ for which: a) $T_t f \geq 0$ when $f \geq 0$; b) $\|T_t f\| \leq \|f\|$; c) all functions from the domain of definition of the infinitesimal operator satisfy the condition $\mathcal{U}$.*

Let $w(z)$ be a function defining a conformal mapping of the domain E onto the unit disk. Denote by P the set of all functions of the form

$$F(z) = \sum_{\gamma \in \Gamma} k_\gamma \arg\left(1 - \frac{w(z)}{w(\gamma)}\right) + F_0(z),$$

where k_γ are constants, and $F_0(z)$ is a function continuous in the closed domain $\overline{E} \cup L$. Each function $F \in P$ is naturally extended to the set $E^* \cup \Pi$. Denote by P_Ω the set of all functions $F \in P$ for which the value $F(\alpha)$ is constant on each class ω of Ω.

Theorem 3. *For the semigroup described in Theorem 2, the space $\widetilde{B}_0 = B_0(\tilde{\mathcal{E}})$ consists of all functions $F \in P_\Omega$ satisfying the conditions: $(F - F(\omega), v_\omega) - c_\omega F(\omega) = 0$ for all $\omega \in \Omega'$. The domain of definition of the infinitesimal operator consists of all functions F satisfying the condition $\mathcal{U}$ and such that $\mathfrak{A}F \in B_0$.*

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Note: Figure translations are in progress. See original paper for figures.