ON SOME CONDITIONS FOR MUTUAL CONVERGENCE OF FUNCTIONS

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ON SOME CONDITIONS FOR MUTUAL CONVERGENCE OF FUNCTIONS

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For a continuously varying scalar argument $t \geq 0$, consider two systems of functions: $d_i(t)$ and $c_j(t)$. The index i takes integer values from zero to p, and the index j takes integer values from zero to q $(p \geq 0,\ q \geq 0)$. Let

$$D_i(t) = td_i(t) + D_i(0); \qquad C_j(t) = tc_j(t) + C_j(0). \tag{1}$$

The functions $d_i(t)$ and $c_j(t)$ satisfy the following conditions: there exists a nonnegative constant $a < \infty$ and a nonnegative function $b(t)$, bounded for every t, such that

$$\lim_{t\to\infty} \frac{b(t)}{t} = 0, \tag{2}$$

and, for $\xi \geq 0,\ s \geq 0$, the relations

$$\begin{aligned} |D_i(\xi + s) - D_i(\xi)| &\leq as + b(\xi + s), \\ |C_j(\xi + s) - C_j(\xi)| &\leq as + b(\xi + s). \end{aligned} \tag{3}$$

The functions $d_0(t)$ and $c_0(t)$ are the principal ones, and the remaining functions are auxiliary. The functions $d_0(t)$ and $c_0(t)$, as well as the functions $D_0(t)$ and $C_0(t)$, will be called **mutually convergent** if

$$\lim_{t\to\infty} \frac{E(t)}{t} = 0, \qquad E(t) = D_0(t) - C_0(t). \tag{4}$$

In a number of cases, the investigation of the mutual convergence of the functions $d_0(t), c_0(t)$ $(D_0(t), C_0(t))$ is expediently carried out with the aid of an auxiliary system of functions. Below we consider the question of necessary and sufficient conditions for mutual convergence in which an auxiliary system of functions is used, as well as the question of estimating the rate of convergence for this case.

Introduce into consideration a nonnegative function $P(t)$, bounded on every finite interval, for which

$$\lim_{t\to\infty} \frac{P(t)}{t} = 0. \tag{5}$$

Let $0 \le h \le t$. The sets $M(t,h)$ and $m(t,h)$ are introduced. The set $M(t,h)$ contains the indices i $(i = 0, 1, \dots, p)$, and $i \in M(t,h)$ if and only if there exists a t' $(t - h \le t' \le t)$ such that

$$|D_0(t') - D_i(t')| \le P(t'). \tag{6}$$

The set $m(t,h)$ contains the indices j $(j = 0, 1, \dots, q)$, and $j \in m(t,h)$ if there exists a t' $(t - h \le t' \le t)$ such that

$$|C_0(t') - C_j(t')| \le P(t'). \tag{7}$$

The sets are not empty, since always $i = 0$ belongs to $M(t,h)$, and $j = 0$ belongs to $m(t,h)$.

Let us introduce the following functions of the arguments t, h and t, z $(0 \le h \le t,\ 0 \le z \le t)$:

$$A_{i,j}(t,h) = D_i(t) - D_0(t-h) - C_j(t) + C_0(t-h), \tag{8}$$

$$G(t,h) = \min_{i\in M(t,h)} \min_{j\in m(t,h)} A_{i,j}(t,h), \tag{9}$$

$$g(t,h) = \min_{i\in M(t,h)} \min_{j\in m(t,h)} (-A_{i,j}(t,h)), \tag{10}$$

$$G_1(t,z) = \max_{0\le h\le z} G(t,h), \tag{11}$$

$$g_1(t,z) = \max_{0\le h\le z} g(t,h). \tag{12}$$

Theorem 1. For the mutual convergence of the functions $D_0(t)$ and $C_0(t)$, it is necessary and sufficient that

$$\lim_{t\to\infty} \frac{G_1(t,t)}{t} = \lim_{t\to\infty} \frac{g_1(t,t)}{t} = 0. \tag{13}$$

Let us note that, in order to establish the convergence of the functions $G_1(t,t)$ and $g_1(t,t)$, they need be estimated only from above, since the lower limits of the ratios $G_1(t,t)/t$ and $g_1(t,t)/t$ are equal to zero in all cases.

The subsequent arguments are preparatory to formulating an estimate of the rate of convergence.

Introduce the following preliminary estimates for $0 \le z \le t$:

$$\begin{gathered}
G_2(t,z) = \max_{t-z<t'\le t} G_1(t', t'-t+z), \qquad g_2(t,z) = \max_{t-z<t'\le t} g_1(t', t'-t+z), \\
b_1(t,z) = \max_{t-z<t'\le t} b(t'), \qquad P_1(t,z) = \max_{t-z<t'\le t} P(t'), \\
G_3(t,z) = G_2(t,z) + 2P_1(t,z) + 2b_1(t,z), \\
g_3(t,z) = g_2(t,z) + 2P_1(t,z) + 2b_1(t,z).
\end{gathered} \tag{14}$$

Let us note that $G_3(t,z) \ge 0$ and $g_3(t,z) \ge 0$.

A sequence of coefficients A_k is introduced, defined by the following recurrence relations:

$$A_0 = 0, \qquad A_{k+1} = \left(\frac{k+1}{k+2}\right)^{(2k+3)/k} \frac{k+3}{k+2} \cdot 2^{-1/k} +$$

$$+ 2^{1/(k+2)} \left(\frac{k+2}{k+1}\right)^{(k+1)/(k+2)} \left(A_k + 2\frac{k+1}{k+2}\right)^{(k+1)/(k+2)}. \tag{15}$$

The sets $\overline{M}_D(t,h)$, $\underline{M}_D(t,h)$, $\overline{m}_C(t,h)$, $\underline{m}_C(t,h)$ are introduced. To the set $\overline{M}_D(t,h)$ ($\underline{M}_D(t,h)$) we assign those indices $i \in M(t,h)$ for which equality (9) (equality (10)) is realized. To the set $\overline{m}_C(t,h)$ ($\underline{m}_C(t,h)$) we assign those indices $j \in m(t,h)$ for which equality (9) (equality (10)) is realized.

Choose arbitrarily one element from the set $\overline{M}_D(t,h)$, and denote it by $\overline{i}_D(t,h)$. Into the set $\overline{M}_{D,1}(t,z)$ we include all those indices i ($i = 0, 1, \dots, p$) which, for at least one aggregate of arguments t', h ($0 \le t-z \le t'-h \le t' \le t$), are indices $\overline{i}_D(t',h)$. The number of elements in the set $\overline{M}_{D,1}(t,z)$ will be denoted by $\overline{p}(t,z)$.

Analogously to the way in which, for the aggregate of sets $\overline{M}_D(t',h)$ for $t-z \le t'-h \le t' \le t$, the quantity $\overline{p}(t,z)$ was put into correspondence, to the aggregates of the sets $\underline{M}_D(t',h)$, $\overline{m}_C(t',h)$, $\underline{m}_C(t',h)$ we put into correspondence the quantities $\underline{p}(t,z)$, $\overline{q}(t,z)$, $\underline{q}(t,z)$.

Introduce the numbers $\overline{R}(t,z)$ and $\underline{R}(t,z)$:

$$\overline{R}(t,z) = \overline{p}(t,z) + \overline{q}(t,z), \qquad \underline{R}(t,z) = \underline{p}(t,z) + \underline{q}(t,z). \tag{16}$$

Introduce the quantities $\bar{r}(t, z)$ and $\underline{r}(t, z)$ for $0 \le z \le t$.

The quantity $\bar{r}(t, z)$ is the greatest of the integers k having the property that, for all t', z' $(t - z \le t' - z' \le t' \le t)$ for which $\bar{R}(t', z') \le k$, the relation

$$E(t') - E(t' - z') \le G_3(t, z) \tag{17}$$

is satisfied.

It can be shown that $\bar{r}(t, z) \ge 2$.

The quantity $\underline{r}(t, z)$ is the greatest of the integers k having the property that, for all such t', z' $(t - z \le t' - z' \le t' \le t)$ for which $\underline{R}(t, z) \le k$, the relation

$$E(t' - z') - E(t') \le g_3(t, z). \tag{18}$$

is satisfied.

Introduce the quantities $\bar{k}(t, z)$ and $\underline{k}(t, z)$:

$$\bar{k}(t, z) = \bar{R}(t, z) - \bar{r}(t, z), \qquad \underline{k}(t, z) = \underline{R}(t, z) - \underline{r}(t, z). \tag{19}$$

The quantities $\bar{k}(t, z)$ and $\underline{k}(t, z)$ are determined uniquely only for a prescribed rule for choosing the element $i_D(t, h)$ from the set $\bar{M}_D(t, h)$ and analogous elements from the sets $\underline{M}_D(t, h)$, $\bar{m}_C(t, h)$, and $\underline{m}_C(t, h)$. We shall assume that $\bar{k}^*(t, z)$ is the minimum of $\bar{k}(t, z)$, and $\underline{k}^*(t, z)$ the minimum of $\underline{k}(t, z)$, over some class of selection rules.

The estimates

$$0 \le \bar{k}^*(t, z) \le p + q + 1, \qquad 0 \le \underline{k}^*(t, z) \le p + q + 1. \tag{20}$$

hold.

Theorem 2. The following estimate holds for the increment of the function $E(t)$ on the interval $[t - z, t]$:

$$-2b_1(t, z) - g_3(t, z) - A_{\underline{k}} \cdot 2a \left(\frac{g_3(t, z)}{2az} \right)^{1/(\underline{k}+1)} z \le$$

$$\le E(t) - E(t - z) \le 2b_1(t, z) + G_3(t, z) + A_{\bar{k}} \cdot 2a \left(\frac{G_3(t, z)}{2az} \right)^{1/(\bar{k}+1)} z, \tag{21}$$

where $\bar{k} = \bar{k}^*(t, z)$, $\underline{k} = \underline{k}^*(t, z)$.

If in the conditions of Theorem 2 we put $z = t$, then a majorized estimate of the rate of convergence of the quantity $E(t)/t$ to zero is obtained (under the conditions in which Theorem 1 holds). In the case when the function $G_3(t+z, z)$ $(g_3(t + z, z))$ is decreasing in t, more exact estimates of the rate of convergence can also be obtained with the aid of Theorem 2. In particular, if $G_3(t+z,z) \le Bt^{-\gamma}$, $\gamma \ge 0$, then

$$\frac{E(t)}{t} \le \frac{\alpha(B,\gamma)}{t} + 2a\left(\frac{B}{2at}\right)^{(1+\gamma)[\bar{k}^*(t,t)+1]} \beta(B,\bar{k}^*(t,t)), \tag{22}$$

where $\alpha(B,\gamma)$, $\beta(B,k)$ are functions of the corresponding arguments.

The proofs of Theorem 2 and Theorem 1 (as regards the sufficiency of the convergence conditions) are based on the following lemmas.

Lemma 1. If a nonnegative function $v(s)$ $(s \ge 0)$ majorizes the quantity $E(t') - E(t'-s)$ $(s \le t')$ on all intervals $[t'-s,t']$ contained in the interval $[t-z,t]$, for which $\bar{R}(t',s) \le k$, then the function $u(s)$, defined by the equations

$$u(s) \ge \min_{0\le\xi\le s-\delta} \max\{f(s,\xi); u(\xi)+v(s-\xi)\}, \qquad s > \delta,$$

$$u(s) \ge 2b_1(t,z) + 2as, \qquad s \le \delta;$$

$$f(s,\xi) = \min\{2b_1(t,z)+2as;\ 4a\xi + 2b_1(t,z) + G_3(t,z)\},$$

$$0 < \delta \le G_3(t,z)/2a,$$

majorizes the quantity $E(t') - E(t'-s)$ on all intervals $[t'-s,t']$ contained in the interval $[t-z,t]$, for which $\bar{R}(t',s) \le k+1$.

Lemma 2. If $\bar{R}(t,z) = 2$, then for t', z' $(0 \le t-z \le t'-z' \le t' \le t)$ relation (17) is satisfied.

Let us make some remarks on the practical applicability of Theorems 1 and 2. The theorems are convenient to use, in particular, if $D_0(t)$ and $C_0(t)$ are given functions of $D_i(t)$ $(i \ge 1)$ and $C_j(t)$ $(j \ge 1)$. When a new function $D_i(t)$ $(C_j(t))$ is added to the already existing ones, the quantities $G(t,h)$ and $g(t,h)$ can either decrease or remain unchanged. Therefore, as the number of functions in the auxiliary systems is increased, the estimation from above of the quantities $G_1(t,t)$ and $g_1(t,t)$ is facilitated.

As an example, let us give an application of Theorems 1 and 2 to proving convergence and estimating its rate for the Brown iterative process, well known in game theory.

Suppose there is a two-person zero-sum matrix game with payoff matrix $\|a_{i,j}\|$, in which the maximum modulus of an entry is equal to a, the number of rows is equal to p (they correspond to the strategies D_i of player D, who maximizes the payoff), and the number of columns is equal to q (they correspond to the strategies C_j of player C, who minimizes the payoff).

Let, for integer values of t, the function $D_i(t)/t$ $(i = 1,\dots,p)$ be the payoff of player D if he uses the pure strategy D_i, while player C uses the mixed strategy "accumulated" over t steps of the Brown process; and let the function $C_j(t)/t$ $(j = 1,\dots,q)$ be the payoff of player D if he uses the mixed strategy "accumulated" over t steps of the Brown process, while player C uses the pure strategy C_j,

$$D_0(t) = \max_i D_i(t), \qquad C_0(t) = \min_j C_j(t),$$

and, for noninteger values of the argument, let all functions vary linearly and continuously.

It can be shown that in this case, for $P(t) = 0$, the relations

$$G_1(t,t) = G_3(t,t) \leq 2a, \qquad g_1(t,t) = g_3(t,t) = 0, \qquad \bar{k}^*(t,t) \leq p + q - 3.$$

hold.

Hence, on the basis of Theorem 1 there follows J. Robinson' s theorem [1] on the convergence of the Brown process, and on the basis of Theorem 2 one obtains an estimate of its rate of convergence, better in the coefficient of $t^{-1(p+q-2)}$ than in the paper of H. Shapiro [2]. However, Theorems 1 and 2 can, in particular, be applied to establishing the fact of convergence not only of the Brown process, but also of a sufficiently broad class of iterative processes for determining the value of matrix games.

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[2] H. N. Shapiro, Comm. Pure and Appl. Math., **11**, 4, 588 (1958); Russian transl. in the collection *Matrix Games*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.

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