ASYMPTOTIC SPLITTING OF THE EQUATIONS OF A LINEAR AUTOMATIC CONTROL SYSTEM

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Abstract

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ASYMPTOTIC SPLITTING OF THE EQUATIONS OF A LINEAR AUTOMATIC CONTROL SYSTEM

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1. The motion of many automatic control systems is described by a system of differential equations which, in vector-matrix form, has the form

$$A(t)\frac{dx}{dt} = B(t)x + f(t). \tag{1}$$

The solution of such systems, or their investigation in view of the variability of the coefficients, is associated with well-known difficulties that increase with the order of the system. Below we set out a method of asymptotic splitting of equations that makes it possible to simplify the problem of investigating linear automatic control systems by transforming the original system of linear differential equations into independent subsystems of lower order.

In what follows, instead of system (1), we shall consider the following system of equations containing a certain parameter ε:

$$A(\tau,\varepsilon)dx/dt = B(\tau,\varepsilon)x + f(t,\tau,\varepsilon) \quad (\tau = \varepsilon t). \tag{2}$$

Here $A(\tau,\varepsilon)$, $B(\tau,\varepsilon)$ are square matrices of order n, admitting expansions in powers of ε:

$$A(\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k A_k(\tau), \qquad B(\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k B_k(\tau) \quad (0 \le \tau \le L). \tag{3}$$

If $A_k(\tau) = B_k(\tau) = 0$ $(k = 1, 2, ...)$, and $\varepsilon = 1$, then system (2) takes the form of system (1).

Under the assumption that $f(t,\tau,\varepsilon)$ is a vector function of the form

$$\sum_{k=0}^{\infty} \varepsilon^k p_k(\tau) e^{i\theta(t,\varepsilon)}$$

(in particular, zero), where $d\theta/dt = k(\tau)$, and $k(\tau)$, $p_k(\tau)$ have on the segment $[0, L]$ a sufficient number of derivatives with respect to τ, it was shown in papers ([1−6]) that there exist transformations leading to an asymptotic splitting of system (2) into independent subsystems of lower order, and the corresponding methods of constructing a formal process for such a splitting were indicated.

In the present note we consider the general case in which $f(t, \tau, \varepsilon)$ is an arbitrary continuous vector function, regular with respect to ε in a neighborhood of the point $\varepsilon = 0$.

As in the works cited above, we use asymptotic methods developed by N. M. Krylov and N. N. Bogolyubov ([7,8]).

2. Let a square matrix $u(\tau)$, whose eigenvalues are divided into p groups

$$\lambda_1^{(\sigma)}, \dots, \lambda_{k_\sigma}^{(\sigma)} \left(\sigma = 1, \dots, p;\ \sum_{\sigma=1}^{p} k_\sigma = n\right),$$

be such that on the segment $[0, L]$ the conditions

$$\left|\lambda_i^{(\sigma)}(\tau) - \lambda_j^{(s)}(\tau)\right| \geq c > 0 \quad (\sigma \neq s;\ i = 1, \dots, k_\sigma;\ j = 1, \dots, k_s), \tag{4}$$

are satisfied.

can be represented in the form ([9, 10])

$$u(\tau) = \sum_{\sigma=1}^{p} K_\sigma(\tau) \Lambda_\sigma(\tau) M_\sigma(\tau), \tag{5}$$

where $K_\sigma, \Lambda_\sigma, M_\sigma$ are matrices of types, respectively, $n \times k_\sigma$, $k_\sigma \times k_\sigma$, $k_\sigma \times n$, satisfying the equalities

$$M_\sigma K_s = \begin{cases} E_{k_\sigma} & (s = \sigma), \\ 0 & (s \neq \sigma) \end{cases} \tag{6}$$

(E_r is the identity matrix of order r). Introducing the coagulated matrices

$$K = (K_1 \dots K_p), \qquad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ \cdot & \cdot \\ 0 & \Lambda_p \end{pmatrix}, \qquad M = \begin{pmatrix} M_1 \\ \cdot \\ \dot{M}_p \end{pmatrix},$$

we shall also have

$$u = K\Lambda M, \qquad MK = KM = E_n. \tag{7}$$

Remark 1. As the matrix K_σ one may take any matrix composed of k_σ linearly independent linear combinations of the columns of the matrix

$$\Delta_\sigma(u) = \prod_{\substack{s=1\\ s\neq\sigma}}^{p} \prod_{j=1}^{k_s} (u - \lambda_j^{(s)} E_n),$$

and, in particular, of k_σ linearly independent columns of this matrix. Knowing the matrix K, it is easy to determine M and Λ, using relations (7).

Remark 2. The eigenvalues of the matrix Λ_σ are the eigenvalues of the matrix u included in group σ.

Remark 3. Using the arbitrariness that exists in constructing the matrix K_σ, one can always ensure the differentiability of K_σ as many times as the matrix u is differentiable. Thus, for example, if on the segment $[0, L]$ the linear independence of some fixed k_σ columns of the matrix $\Delta_\sigma(u)$ is not violated, then it is sufficient to take as K_σ the matrix consisting of these k_σ columns.

Taking this into account, in what follows we shall assume that K, and consequently also Λ and M, have as many derivatives with respect to τ as the matrix u has.

3. Theorem. If on the segment $[0, L]$ the matrices $A_k(\tau), B_k(\tau)$ $(k = 0, 1, 2, \dots)$ have the required number of derivatives with respect to τ, and $A_0(\tau)$, moreover, is a nonsingular matrix, then, assuming that the eigenvalues of the matrix $u(\tau) = A_0^{-1}(\tau)B_0(\tau)$ are divided into p groups under condition (4), the formal solution of system (2) can be represented as follows:

$$x = \sum_{\sigma=1}^{p} \widetilde{K}_\sigma(\tau, \varepsilon) y_\sigma, \tag{8}$$

where

$$dy_\sigma/dt = \widetilde{\Lambda}_\sigma(\tau, \varepsilon) y_\sigma + \widetilde{M}_\sigma(\tau, \varepsilon) R(\tau, \varepsilon) f(t, \tau, \varepsilon) \qquad (\sigma = 1, \dots, p). \tag{9}$$

Here $\widetilde{K}_\sigma, \widetilde{\Lambda}_\sigma, \widetilde{M}_\sigma, R$ are matrices of types, respectively, $n \times k_\sigma$, $k_\sigma \times k_\sigma$, $k_\sigma \times n$, $n \times n$, representable by formal series

$$\widetilde{K}_\sigma(\tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k K_\sigma^{[k]}(\tau), \qquad \widetilde{\Lambda}_\sigma(\tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \Lambda_\sigma^{[k]}(\tau), \tag{10}$$

$$\widetilde{M}_\sigma(\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k M_\sigma^{[k]}(\tau), \qquad R(\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k R_k(\tau).$$

Substituting formulas (8), (9), which determine the vector x, into system (2) and separating in the resulting identity the coefficients of y_σ ($\sigma = 1, \dots, p$) and the free term, we shall have

$$\varepsilon A(\tau,\varepsilon)\, d\widetilde{K}_\sigma(\tau,\varepsilon)/d\tau + A(\tau,\varepsilon)\widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{\Lambda}_\sigma(\tau,\varepsilon) = B(\tau,\varepsilon)\widetilde{K}_\sigma(\tau,\varepsilon)$$
$$(\sigma = 1, \dots, p); \tag{11}$$

$$\left[A(\tau,\varepsilon)\sum_{\sigma=1}^{p} \widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{M}_\sigma(\tau,\varepsilon)R(\tau,\varepsilon) - E_n\right] f(t,\tau,\varepsilon) = 0. \tag{12}$$

In equality (11) we substitute the expansions (3) and (10) and equate the coefficients of like powers of ε. We obtain

$$uK_\sigma^{[0]} = K_\sigma^{[0]}\Lambda_\sigma^{[0]},$$

$$uK_\sigma^{[k]} = K_\sigma^{[k]}\Lambda_\sigma^{[0]} + K_\sigma^{[0]}\Lambda_\sigma^{[k]} + D_\sigma^{[k-1]} \tag{13}$$

$$(k = 1, 2, \dots;\ \sigma = 1, \dots, p).$$

By $D_\sigma^{[k-1]}$ is denoted an expression depending only on quantities up to the $(k-1)$-st approximation inclusively.

Put

$$K_\sigma^{[0]}(\tau) \equiv K_\sigma(\tau), \qquad \Lambda_\sigma^{[0]}(\tau) \equiv \Lambda_\sigma(\tau). \tag{14}$$

Then the first equality (13) is satisfied identically.

Suppose that $K_\sigma^{[0]}, \Lambda_\sigma^{[0]}, \dots, K_\sigma^{[k-1]}, \Lambda_\sigma^{[k-1]}$ have already been found. We determine $K_\sigma^{[k]}$ and $\Lambda_\sigma^{[k]}$. From the $(k+1)$-st equality (13) we obtain:

$$\Lambda Q_\sigma^{[k]} = Q_\sigma^{[k]}\Lambda_\sigma + MK_\sigma\Lambda_\sigma^{[k]} + MD_\sigma^{[k-1]}, \tag{15}$$

where

$$Q_\sigma^{[k]} = MK_\sigma^{[k]}. \tag{16}$$

Since Λ has a quasidiagonal structure, equality (15) splits into the following p independent matrix equalities:

$$\Lambda_\sigma Q_{\sigma\sigma}^{[k]} = Q_{\sigma\sigma}^{[k]} \Lambda_\sigma + \Lambda_\sigma^{[k]} + M_\sigma D_\sigma^{[k-1]}, \tag{17}$$

$$\Lambda_s Q_{s\sigma}^{[k]} = Q_{s\sigma}^{[k]} \Lambda_\sigma + M_{sD_\sigma}^{[k-1]} \qquad (s \neq \sigma), \tag{18}$$

where $Q_{s\sigma}^{[k]} = M_{sK_\sigma}^{[k]}$ $(s = 1, \dots, p)$ are submatrices of the matrix $Q_\sigma^{[k]}$.

From (17) we directly find

$$\Lambda_\sigma^{[k]} = \Lambda_\sigma Q_{\sigma\sigma}^{[k]} - Q_{\sigma\sigma}^{[k]} \Lambda_\sigma - M_\sigma D_\sigma^{[k-1]}. \tag{19}$$

Here $Q_{\sigma\sigma}^{[k]}$ is an arbitrary square matrix of order k_σ, differentiable a sufficient number of times. In particular, one may take $Q_{\sigma\sigma}^{[k]} = 0$.

The equalities (18) uniquely determine $Q_{s\sigma}^{[k]}$ $(s = 1, \dots, p;\ s \neq \sigma)$, since Λ_s and Λ_σ have no common eigenvalues.

Having determined from (18) $Q_{s\sigma}^{[k]}$ $(s \neq \sigma)$ and having specified the matrix $Q_{\sigma\sigma}^{[k]}$, we shall have $Q_\sigma^{[k]}$, after which it is easy to obtain $K_\sigma^{[k]}$ by the formula

$$K_\sigma^{[k]} = K Q_\sigma^{[k]}. \tag{20}$$

Thus, with the aid of the recurrent formulas (19), (18), (20), one can successively determine

$$\Lambda_\sigma^{[1]}, K_\sigma^{[1]}, \Lambda_\sigma^{[2]}, K_\sigma^{[2]}, \dots .$$

Now on the construction of the matrices $\widetilde{M}_\sigma$ and R. Equality (12) can be represented in the form

$$\left[A(\tau,\varepsilon)\widetilde{K}(\tau,\varepsilon)\,\widetilde{M}(\tau,\varepsilon), R(\tau,\varepsilon) - E_n\right] f(t,\tau,\varepsilon) = 0, \tag{21}$$

where

$$\widetilde{K} = \sum_{k=0}^{\infty} \varepsilon^k K^{[k]}, \qquad \widetilde{M} = \sum_{k=0}^{\infty} \varepsilon^k M^{[k]},$$

$$K^{[k]} = (K_1^{[k]} \dots K_p^{[k]}), \qquad M^{[k]} = \begin{pmatrix} M_1^{[k]} \\ \vdots \\ M_p^{[k]} \end{pmatrix}. \tag{22}$$

It is easy to verify that, if the terms of the formal series $\widetilde{M}$ and R are determined by the recurrence formulas

$$M^{[0]} = M, \qquad M^{[k]} = -M \sum_{i=1}^{k} K^{[i]} M^{[k-i]} \quad (k = 1, 2, \ldots); \tag{23}$$

$$R_0 = A_0^{-1}; \qquad R_k = -A_0^{-1} \sum_{i=1}^{k} A_{iR_{k-i}} \quad (k = 1, 2, \ldots), \tag{24}$$

then the equalities

$$K(\tau, \varepsilon) \cdot \widetilde{M}(\tau, \varepsilon) = E_n, \tag{25}$$

$$A(\tau, \varepsilon) \cdot R(\tau, \varepsilon) = E_n, \tag{26}$$

will hold, and consequently equality (12) will also hold.

4. Let $x_m(t, \varepsilon)$ denote the vector determined by equalities (8) and (9), if in formulas (10) we restrict ourselves to terms of order no higher than m with respect to ε.

Let

$$x\big|_{t=t_1} = x_m\big|_{t=t_1}.$$

Then there exists an $\varepsilon_0 > 0$ such that, for certain constants c_m and ε_1 ($\varepsilon_1 \leqslant \varepsilon_0$), on the segment $[t_1, t_2]$, $t_1, t_2 \in [0, L]$, the estimate

$$\|x - x_m\| \leqslant c_m \varepsilon^{m-1} \quad (\varepsilon < \varepsilon_1,\ t \in [t_1, t_2]) \tag{27}$$

holds.

If, in addition to the assumptions made above, all the eigenvalues of the Hermitian matrix $\frac{1}{2}(\Lambda + \Lambda^*)$ are nonpositive, then estimate (27) holds for $0 \leqslant t \leqslant L/\varepsilon$. Thus, the formal solution x_m is asymptotic in character.

Remark. The matrices $\widetilde{K}_\sigma$ and $\widetilde{\Lambda}_\sigma$ do not depend on $f(t, \tau, \varepsilon)$. Therefore, to obtain the solution of the homogeneous system, it is sufficient in the relations given above to put $f(t, \tau, \varepsilon) = 0$.

In the case of the homogeneous system the estimate

$$\|x - x_m\| \leqslant c_m \varepsilon^m$$

holds.

5. It is not difficult to show that the formal solution of the adjoint homogeneous system

$$dz/dt = -B^*(\tau, \varepsilon) z \tag{28}$$

can be represented by the equalities

$$z = \sum_{\sigma=1}^{p} \widetilde{M}_\sigma^* v_\sigma, \qquad dv_\sigma/dt = -\widetilde{\Lambda}_\sigma^* v_\sigma, \tag{29}$$

where $\widetilde{M}_\sigma$ and $\widetilde{\Lambda}_\sigma$ are the matrices appearing in formula (9), constructed under the assumption that $A(\tau, \varepsilon) \equiv E_n$.

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